Isolated horizons — local approach to black holes

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Stationary black holes

Static spherically symmetric solution to vacuum Einstein's equations

Stationary spacetime — there exists a one-parameter group of isometries whose orbits are timelike curves (equivalently, there exist a timelike Killing vector field). Metric components may be chosen to be independent of the time-coordinate.
Static spacetime — stationary and in addition, there exist a spacelike hypersurface which is orthogonal to the orbits of the isometry. There exist time-coordinate, for which metric components are time independent and all time-space components g_{ta} vanish.
Spherically symmetric spacetime — isometry group contains a subgroup isomorphic to $SO(3)$

Schwarzschild solution (to $G_{\mu\nu} = 0$) is:

$$g = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

<u>Features:</u>

[Schwarzschild 1916]

- Describes the spacetime outside of the spherically symmetric star
- Asymptotically flat (as $r \to \infty$, $g_{\mu\nu}$ approaches $\eta_{\mu\nu}$ in spherical coordinates)
- Problem: for $r = 0$ and $r = 2M$ metric components become singular (not a problem for ordinary bodies!)

Birkhoff’s theorem

Any spherically symmetric solution to vacuum Einstein’s equations must be static and asymptotically flat. Therefore, the Schwarzschild metric is the unique vacuum solution with spherical symmetry and there are no time-dependent solutions of this form.

[Birkhoff 1923]

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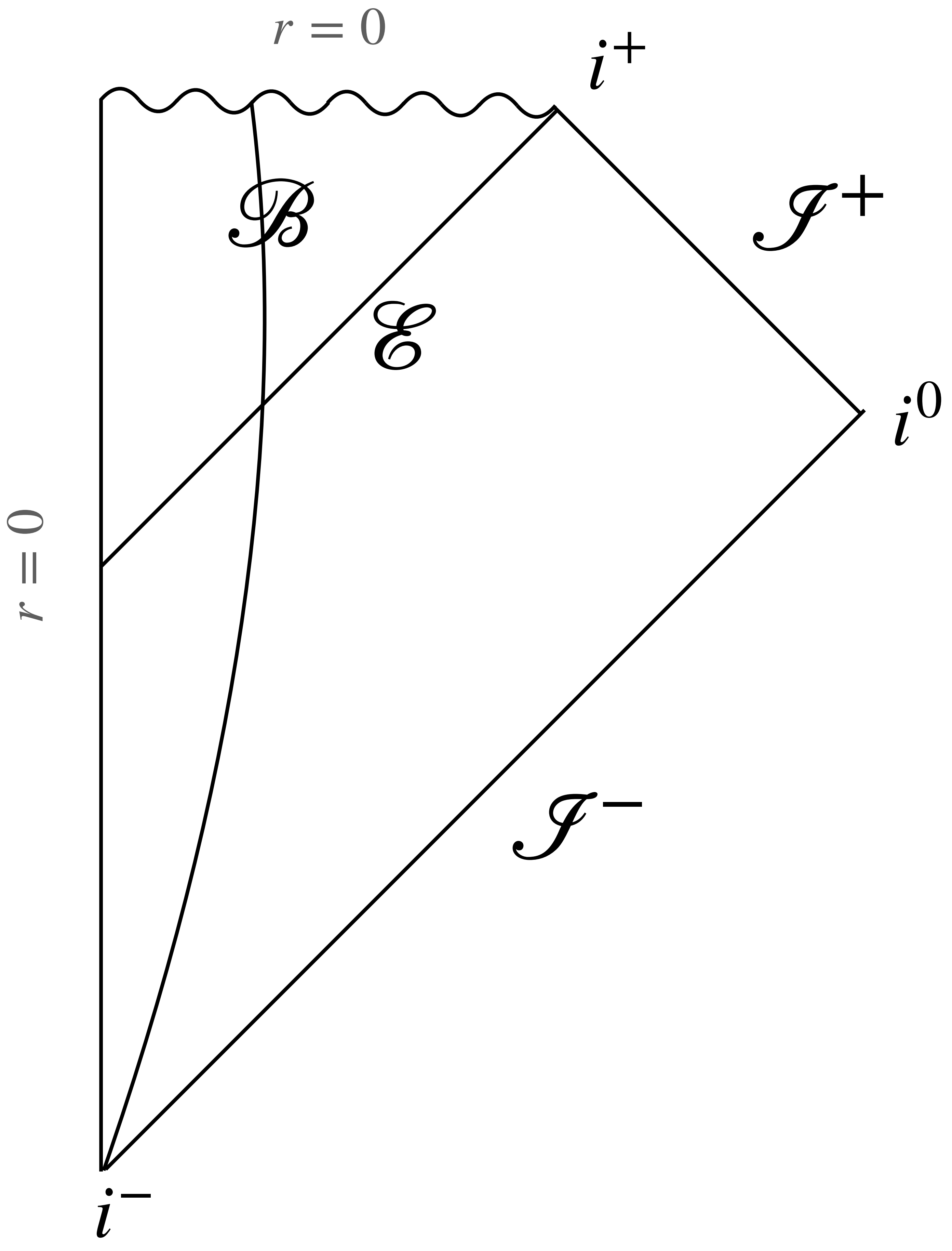
However, spherical symmetry is a strong assumption:

…are black holes physical or just a consequence of unphysical assumptions?

Black holes and event horizons

Notice that on the conformal diagram of the spherical collapse spacetime, region $\mathscr{B}$ of the physical spacetime lies outside of $J^-(\mathscr{I}^+)$, in contrast to Minkowski spacetime, even though $J^-(\mathscr{I}^+)$ is nonsingular.



A black hole region $\mathscr{B}$ of spacetime $(\mathscr{M}, g)$ is defined as

$$\mathscr{B} = \mathscr{M} \setminus J^-(\mathscr{I}^+),$$

the boundary $\partial\mathscr{B}$ of the black hole region is the event horizon $\mathscr{E}$, which is a null 3-dim surface.

Killing horizons

Consider a stationary black hole. There exists a Killing field χ^μ which is null/normal to the horizon of the black hole. If χ^μ does not coincide with the stationary Killing vector field ξ^μ, we obtain it by a linear combination of ξ^μ with rotational Killing vector field ψ^μ:

$$\chi^\mu = \xi^\mu + \Omega_H \psi^\mu$$

In case of a stationary rotating black hole, Kerr black hole, the constant Ω_H is called the angular velocity of the horizon. Since $\chi^\mu \chi_\mu = 0$, it follows that $\nabla^\nu(\chi^\mu \chi_\mu)$ is also normal to the horizon, and consequently there exist a function κ, called surface gravity, such that:

$$\nabla^\nu(\chi^\mu \chi_\mu) = -2\kappa\chi^\nu$$

It can be showed that $\kappa =$ const. on the horizon. The above may be rewritten as:

$$\chi^\mu \nabla_\nu \chi_\mu = -\chi^\mu \nabla_\mu \chi_\nu = -\kappa\chi_\nu$$

which is just the geodesic equation in a non-affine parametrization.

Note: In a static, asymptotically flat spacetime, the surface gravity is the acceleration of the static observer near the horizon, as measured by a static observer at infinity.

No-hair, rigidity and uniqueness theorems

No-hair

Stationary, asymptotically flat black hole solutions to Einstein's equations coupled to electromagnetism that are non-singular outside the event horizon are fully characterized by the following parameters: mass, electric charge and magnetic charge, and angular momentum.

Rigidity

Rotating stationary black hole must be axisymmetric.

Uniqueness

The only possible stationary and axisymmetric black hole solution of the Einstein-Maxwell equations satisfying certain boundary conditions (asymptotic flatness) are Kerr-Newman solutions.

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Stationarity???

Stationary solutions are of special interest, because we expect them to be the end states of gravitational collapse. The alternative might be some sort of oscillating configurations, but oscillations will ultimately be damped as energy lost through the emission of gravitational radiation (black hole mergers, gravitational collapse).

Isolated horizons

Why isolated horizons?

- The laws which are meant to refer just to black holes, are derived assuming that the entire spacetime is stationary. In thermodynamics, by contrast, we only need to assume that the system under consideration is in equilibrium, but not the entire universe.

- The first law of black hole thermodynamics, area A and angular velocity Ω_H are evaluated at the horizon, whereas mass M and angular momentum J are computed at infinity and may include contributions from possible matter fields outside the black hole. It would be more satisfactory to have laws of black hole mechanics which would only involve characteristics of the black hole alone.

- The notion of the event horizon is global and requires a teleological knowledge of the whole spacetime, as it refers to $\mathscr{I}^+$.

What can we do about it?

Drop the requirement that spacetime should admit a stationary Killing field, only ask that the intrinsic horizon geometry be time independent.

Non-expanding horizons

↓

Weakly isolated horizon

↓

Isolated horizons

Non-expanding horizons

Consider 4-dim spacetime that consists of a manifold $\mathscr{M}$ and a metric tensor $g_{\mu\nu}$ of the signature $-+++$.
Let ∇_μ be the torsion free covariant derivative in $\mathscr{M}$, corresponding to $g_{\mu\nu}$:

$$\nabla_\alpha g_{\mu\nu} = 0.$$

We assume that the metric tensor satisfies Einstein equations with cosmological constant

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0.$$

Next, we study a 3-dim null hypersurface in $\mathscr{M}$

$$\mathscr{H} \subset \mathscr{M}.$$

Definition 1. $\mathscr{H}$ is a non-expanding horizon if:

(i) $\mathscr{H}$ contains a slice that intersects each null curve in $\mathscr{H}$ exactly once, in other words $\mathscr{H}$ is of the topology

$$\mathscr{H} = S \times \mathbb{R}$$

and the fibers of the projection

$$\Pi : S \times \mathbb{R} \to S$$

are null curves in $\mathscr{H}$;

(ii) expansion of any null vector field $\ell \in \Gamma\big(T(\mathscr{H})\big)$ vanishes

$$\theta = 0;$$

(iii) stress-energy tensor is such that $-T^a{}_b \ell^b$ is causal and future directed on $\mathscr{H}$.

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Integral curves of ℓ will be called generators of $\mathscr{H}$, S - base space

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From (ii), (iii) and Raychaudhuri equation follows that ℓ is also shear free.

Non-expanding horizons

ℓ is a null vector field $\Rightarrow$ the induced metric g_{ab} on $\mathscr{H}$ is degenerate

$$\ell^a g_{ab} = 0.$$

Since g_{ab} is degenerate, there exist infinitely many derivative operators on $\mathscr{H}$ which are

- torsion-free,
- metric compatible: $\nabla_c g_{ab} = 0$.

However! Its Lie derivative can be uniquely decomposed into expansion and shear

$$\frac{1}{2}\mathscr{L}_\ell g_{ab} = \frac{1}{2}\theta g_{ab} + \sigma_{ab} \quad \Rightarrow \quad \mathscr{L}_\ell g_{ab} = 0$$

Then, for every $X, Y \in \Gamma\big(T(\mathscr{H})\big)$

$$(X^a \nabla_a Y^b)\ell_b = -X^a Y^b \nabla_a \ell_b = 0.$$

This means that spacetime covariant derivative ∇_μ preserves the tangent bundle $T(\mathscr{H})$:

$$X, Y \in \Gamma\big(T(\mathscr{H})\big) \Rightarrow \nabla_X Y \in \Gamma\big(T(\mathscr{H})\big)$$

and endows $\mathscr{H}$ with a covariant derivative ∇_a via the restriction.

The pair (g_{ab}, ∇_a) is called the intrinsic geometry of $\mathscr{H}$.

Non-expanding horizons

We could then define a 1-form ω_a such that

$$\nabla_a \ell^b = \omega_a \ell^b$$

and also surface gravity $\kappa^{(\ell)}$

$$\kappa^{(\ell)} = \omega_a^{(\ell)} \ell^a$$

Since g_{ab} is degenerate, it is the pullback to $\mathscr{H}$ of a Riemannian metric g_{AB} on S:

$$g_{ab} = \Pi^*{}_{ab}{}^{AB} g_{AB}$$

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What follows from such structure?

(i) intrinsic geometry (g_{ab}, ∇_a) of $\mathscr{H}$ determines the pullback R_{ab} of the spacetime Ricci tensor.

(ii) the energy condition (Def. 1 (iii)) together with Raychauduhri equation give

$$R_{ab} \ell^a \ell^b = 0$$

meaning that $R_{ab} \ell^a$ is proportional to ℓ_a. It follows that $R_{ab} \ell^a X^b = 0$ for every $X^a \in \Gamma\big(T(\mathscr{H})\big)$.

(iii) consequences for other components of the curvature — spacetime Weyl tensor

$$\Psi_0 = 0 = \Psi_1$$

the other component Ψ_2 is gauge invariant and

$$d\omega = 2\mathrm{Im}(\Psi_2)\eta$$

(iv) from Einstein equations follows

$$\partial_a \kappa^{(\ell)} = \mathscr{L}_\ell \omega_a$$

Non-expanding horizons — conclusions

3-dim null surface $\mathscr{H}$ in 4-dim spacetime $(\mathscr{M}, g_{\mu\nu})$

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

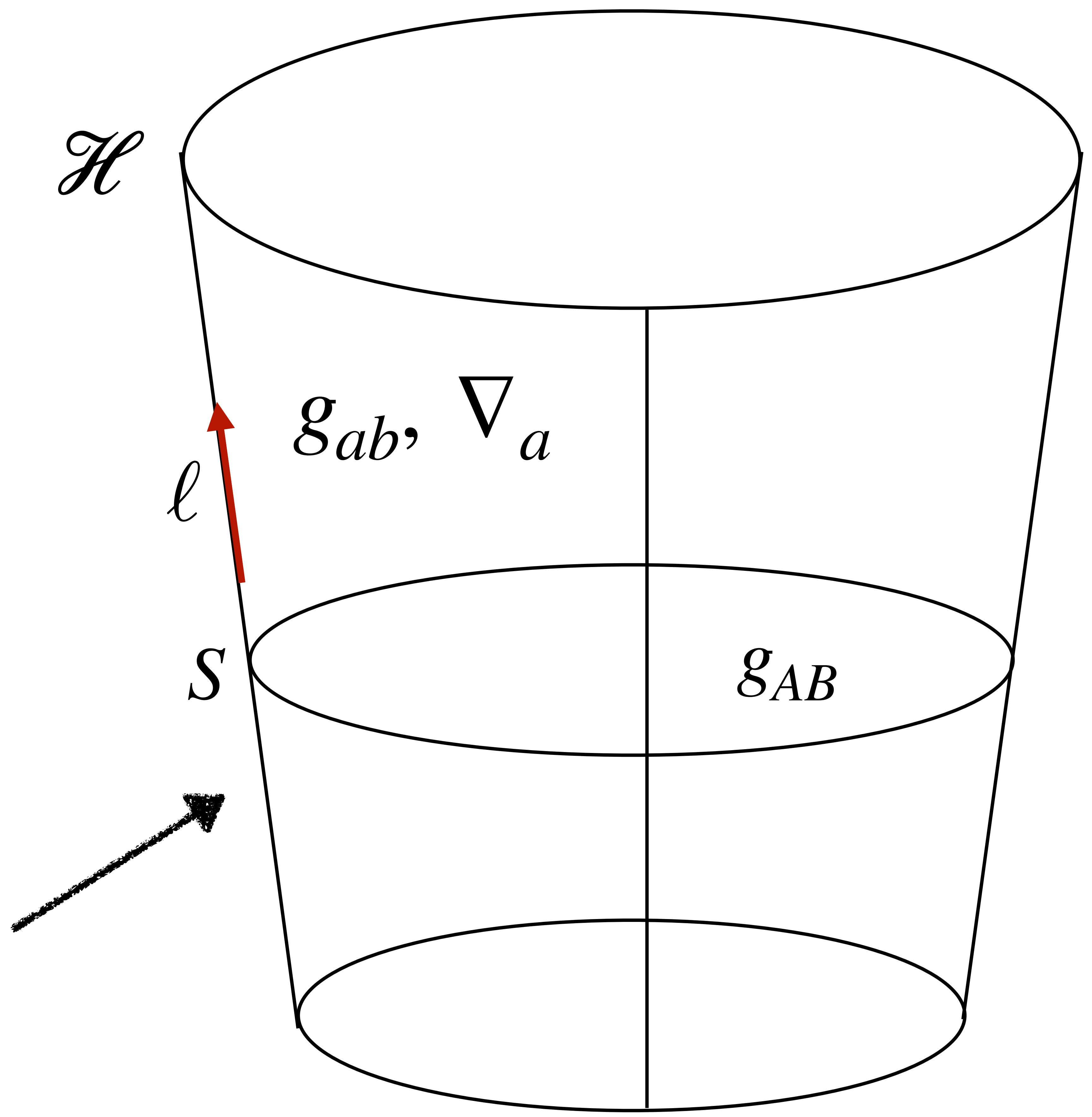
$$\ell^a g_{ab} = 0$$

Rotation potential ω

$$\nabla_a \ell^b = \omega_a \ell^b$$

Surface gravity
$\kappa^{(\ell)} = \omega_a \ell^a$

$$\begin{cases} \mathscr{L}_\ell g_{ab} = 0 \\ \partial_a \kappa^{(\ell)} = \mathscr{L}_\ell \omega_a \\ \Psi_0 = 0 = \Psi_1 \end{cases}$$



Weakly isolated horizon

Definition 2. A weakly isolated horizon $\left(\mathscr{H}, [\ell]\right)$ is a non-expanding horizon, for which the flow of ℓ preserves the rotation 1-form potential

$$\mathscr{L}_\ell \omega_a = 0.$$

It follows that $\partial_a \kappa^{(\ell)} = 0 \Rightarrow \kappa^{(\ell)} = \text{const.}$ (0th law extended to horizons representing local equilibrium).

Covariant derivative ∇_a

- not fully determined by g_{ab} (since g_{ab} is degenerate)
- to specify it completely, we need to know how it acts on a covector field n_a such that $n_a \ell^a \neq 0$, namely

$$S_{ab} := \nabla_a n_b$$

Without loss of generality, we can choose $n = -dv$, where v is a compatible coordinate of ℓ: $\mathscr{L}_\ell v = 1$

- it follows that S_{ab} is symmetric and

$$S_{ab} \ell^b = \omega_a$$

therefore, if we are given (g_{ab}, ω_a) it suffices to provide only the projection S_{AB} of S_{ab} on the sections of $\mathscr{H}$ orthogonal to n_a.

$\Rightarrow$ The geometry of the WIH is fully determined by $(g_{ab}, \omega_a, S_{AB})$

Weakly isolated horizon

What are the constraints when imposing vacuum Einstein's equations on this data?

$$\mathscr{L}_{\ell} S_{AB} = -\kappa^{(\ell)} S_{AB} + \nabla_{(A}\omega_{B)} + \omega_A \omega_B - \frac{1}{2}\mathscr{R}_{AB} + \frac{1}{2}\Lambda g_{AB}$$

where the LHS is „time" derivative of S_{AB}, ∇_A is the induced derivative operator and $\mathscr{R}_{AB}$ the Ricci tensor of S.

Weakly isolated horizons — conclusions

3-dim null surface $\mathscr{H}$ in 4-dim spacetime $(\mathscr{M}, g_{\mu\nu})$

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$$

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Rotation potential ω

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$$\kappa^{(\ell)} = \omega_a \ell^a$$

$$\kappa^{(\ell)} = \text{const}\,.$$

$$\begin{cases} \mathscr{L}_\ell g_{ab} = 0 \\ \mathscr{L}_\ell \omega_a = 0 \end{cases}$$

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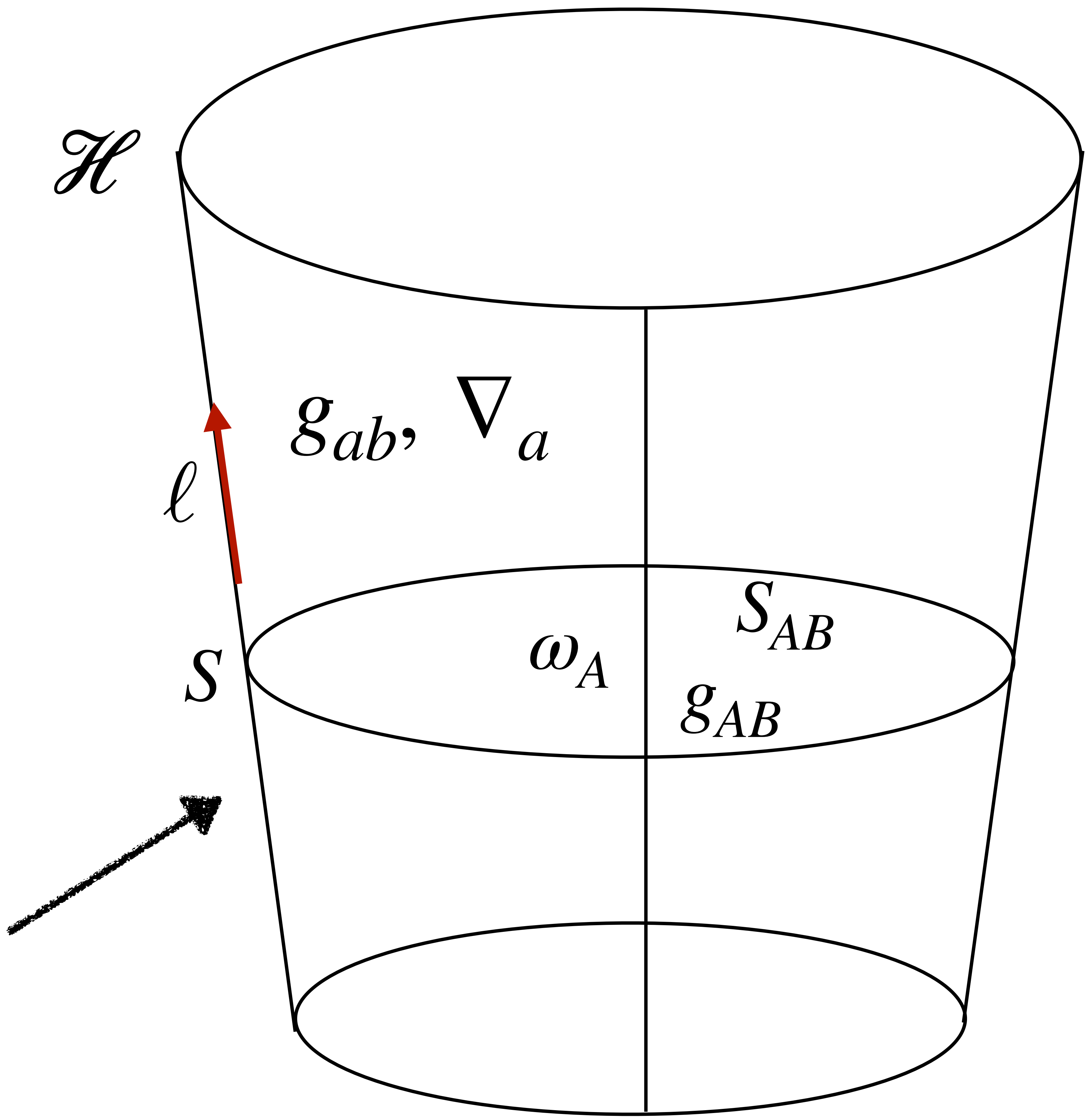

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Isolated horizons

Definition 3. An isolated horizon $\left(\mathscr{H}, [\ell]\right)$ is a weakly isolated horizon, equipped with the null vector field ℓ satisfying:

$$[\mathscr{L}_\ell, \nabla_a] = 0.$$

$\Rightarrow$ Not only g_{ab} and ω_a are „time" independent ($\mathscr{L}_\ell g_{ab} = 0 = \mathscr{L}_\ell \omega_a$) but the entire geometry (g_{ab}, ∇_a), in this sense it is isolated.

Implications:

- time independence of S_{AB}, for non-extremal IH:

$$\mathscr{L}_\ell S_{AB} = 0 \Rightarrow S_{AB} = \frac{1}{\kappa^{(\ell)}}\left(\nabla_{(A}\omega_{B)} + \omega_A\omega_B - \tfrac{1}{2}\mathscr{R}_{AB} + \tfrac{1}{2}\Lambda g_{AB}\right)$$

  which means that it is fully specified by the data on cross section S.
- to determine the geometry of a non-extremal ($\kappa^{(\ell)} \neq 0$) IH we need to fix the cross section S of $\mathscr{H}$, whereas the fields (g_{ab}, ω_a) are such that:
  (i) g_{ab} is a pullback of positive definite metric g_{AB} on S
  (ii) metric g_{ab} is degenerate: $\ell^a g_{ab} = 0$
  (iii) surface gravity is constant: $\omega_a \ell^a = \kappa^{(\ell)} = \text{const.}$
  (iv) extend all fields to $\mathscr{H}$, by requiring that they are Lie-dragged by ℓ: $\mathscr{L}_\ell(\,\cdot\,) = 0.$

Isolated horizons — conclusions

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Surface gravity

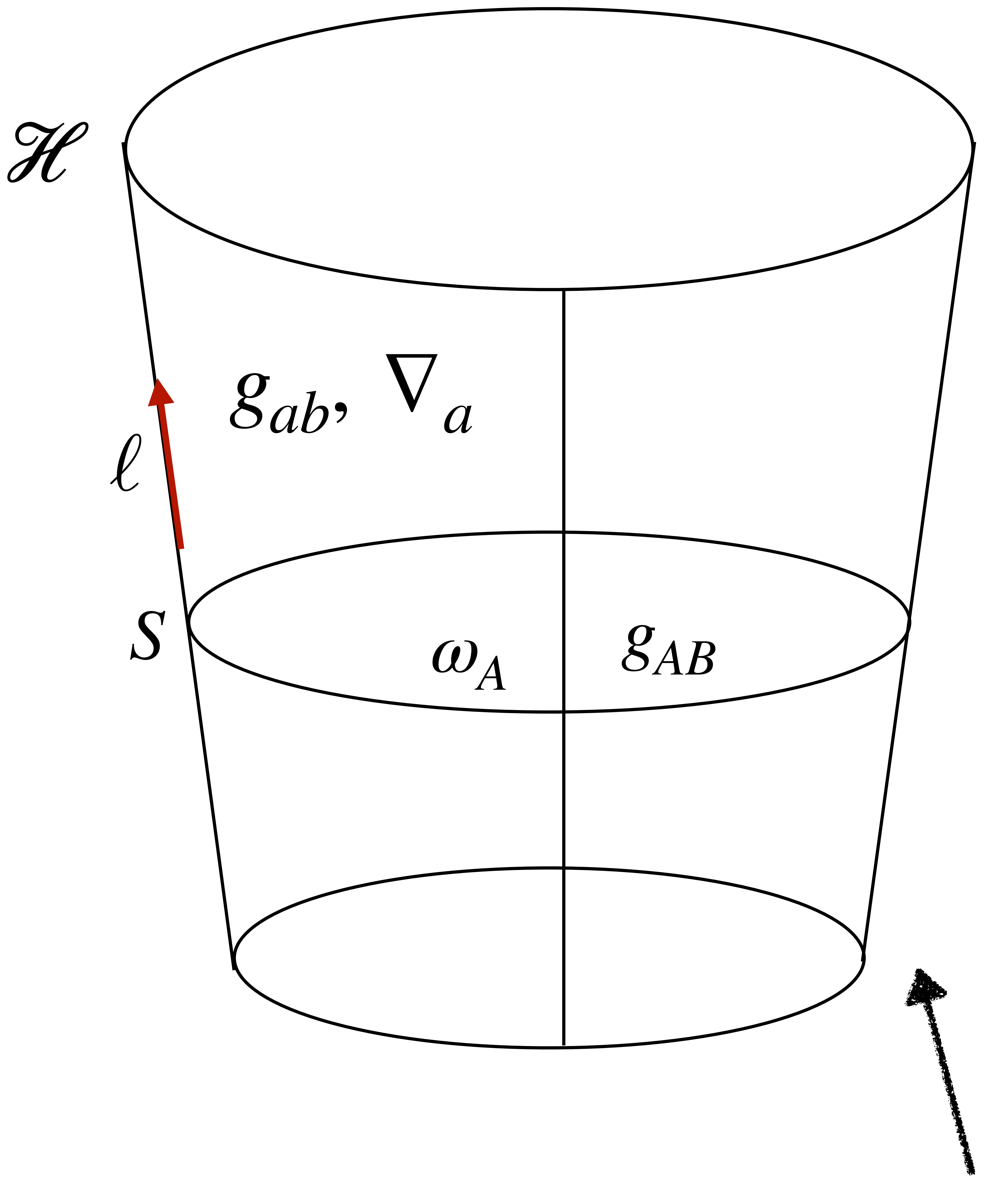
$$\kappa^{(\ell)} = \omega_a \ell^a$$

$$\kappa^{(\ell)} = \text{const} \,.$$

Non-extremality condition

$$\kappa^{(\ell)} \neq 0$$

$$\begin{cases} \mathscr{L}_\ell g_{ab} = 0 \\ [\mathscr{L}_\ell \,, \nabla_\mu] = 0 \\ \mathscr{L}_\ell R_{\mu\nu\alpha\beta} = 0 \end{cases}$$



$$(\omega_A, g_{AB}) \longrightarrow (\omega_a, g_{ab}) \longrightarrow g_{\mu\nu}, \nabla_\mu, R_{\mu\nu\alpha\beta}$$

[Ashtekar, Beetle, Lewandowski 2002]

Weyl tensor in Newman-Penrose components

Spacetime Weyl tensor in the null frame formalism may be expressed by the following complex valued N-P components:

$$\Psi_0 = C_{4141} \qquad \Psi_1 = C_{4341} \qquad \Psi_2 = C_{4123} \qquad \Psi_3 = C_{3432} \qquad \Psi_4 = C_{3232}$$

Four components are constant along null generators of $\mathscr{H}$:

$$D := \ell^a \partial_a$$

$$D\Psi_I = 0, \qquad I = 0,\ 1,\ 2,\ 3$$

Assumption of the stationarity to the second order:

$$D\Psi_4 = 0$$

The components Ψ_0 and Ψ_1 vanish due to the vanishing of expansion and shear of ℓ:

$$\Psi_0 = 0 = \Psi_1$$

The component Ψ_2 may be expressed in terms of the Gaussian curvature K and rotation scalar Ω:

$$\Psi_2 = -\tfrac{1}{2}(K + i\Omega) + \tfrac{1}{6}\Lambda \quad \text{where} \quad \Omega\eta_{AB} = d\omega_{AB}$$

Petrov type D equation

The spacetime Weyl tensor at $\mathscr{H}$ is determined by: (S, g_{AB}, ω_A)

Theorem 1: The possible Petrov types of $\mathscr{H}$ are: I, II, D, III, N, O

$$\Psi_2 = 0 \quad \Leftrightarrow \quad O \quad \Leftrightarrow \quad K = \frac{\Lambda}{3},\ \Omega = 0$$

$\Psi_2 \neq 0 \quad \Leftrightarrow \quad$ generically type II, unless Petrov type D equation is satisfied

We use a null 2-frame:

$$g_{AB} = m_A \bar{m}_B + \bar{m}_A m_B \qquad \eta_{AB} = i(\bar{m}_A m_B - \bar{m}_B m_A)$$

Theorem 2: At $\mathscr{H}$ the spacetime Weyl tensor is of the Petrov type D iff the following two conditions are satisfied:

$$\Psi_2 \neq 0 \quad \text{and} \quad \bar{m}^A \bar{m}^B \nabla_A \nabla_B \Psi_2^{-\frac{1}{3}} = 0$$

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[Lewandowski, Pawłowski 2003 for $\Lambda = 0$]
[DDR, Lewandowski, Pawłowski 2018]

Solutions to the type D equation

Axisymmetric 2-sphere

No hair theorem. The family of axisymmetric solutions to the Petrov type D equation with (or without) cosmological constant defined on a topological sphere can be parametrized by a pair (A, J), that is the area and angular momentum. They can take the following values:

$$\text{for } \Lambda > 0 : J \in \Big(-\infty, \infty \Big) \text{ for } A \in \Big(0, \frac{12\pi}{\Lambda} \Big) \text{ and } |J| \in \Big[0, \frac{A}{16\pi} \sqrt{\frac{\Lambda\, A}{12\pi} - 1} \Big) \text{ for } A \in \Big(\frac{12\pi}{\Lambda}, \infty \Big)$$

$$\text{for } \Lambda < 0 : J \in \Big(-\infty, \infty \Big) \text{ and } A \in \Big(0, \infty \Big)$$

Every solution defines a type D isolated horizon whose intrinsic geometry coincides with the intrinsic geometry of a non-extremal Killing horizon contained in one of the following spacetimes:
(i) Kerr-(anti) de Sitter
(ii) Schwarzschild-(anti) de Sitter
(iii) Near horizon limit spacetime near an extremal horizon contained either in the Kerr-(anti) de Sitter or Schwarzschild-(anti) de Sitter spacetime.

[Lewandowski, Pawłowski 2003 for $\Lambda = 0$]
[DDR, Lewandowski, Pawłowski 2018]

Genus>0 compact 2-surface

Theorem 4. Suppose S is a compact 2-surface of genus>0. The only solution to the Petrov type D equation with cosmological constant Λ are such (g_{AB}, ω_A) that

$$d\omega_{AB} = 0 \qquad \text{and} \qquad K = \text{const} \neq \frac{\Lambda}{3}$$

therefore, non-rotating and of constant Gauss curvature.

[DDR, Kaminski, Lewandowski, Szereszewski 2018]

IH of nontrivial topology

Consider the $\mathscr{H}$ of the structure of the $U(1)$-bundle over 2-manifold diffeomorphic to S_2,

$$\text{then: } \int_{S_2} \Omega\eta = 2\pi\kappa m =: 2\pi n \neq 0$$

where $m \in \mathbb{Z}$ characterizes the $U(1)$-bundle:

Theorem 5. All axisymmetric solutions which for every value of the topological charge m set a 3-dim family that can be parametrized by the area, Kerr and NUT parameters.
Embeddable in Kerr-NUT-(anti) de Sitter spacetimes.

[DDR, Lewandowski, Racz 2020]

[Lewandowski, Ossowski 2020]

If we allow for conical singularities, then

Theorem 6. All axisymmetric solutions set a 4-dim family that can be parametrized by the area, acceleration, Kerr and NUT parameters.
Embeddable in Accelerated Kerr-NUT-(anti) de Sitter spacetimes.

[DDR, Lewandowski, Ossowski 2023]

Thank you!