Bose-Einstein Condensation and Spontaneous Symmetry Breaking

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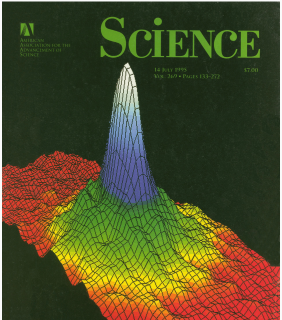
Bose-Einstein Condensation

- **1924 Bose** and **Einstein** discover a completely new type of **phase transition** in an ideal quantum gas.
- **Macroscopic occupation** of a common quantum state at low temperatures.
- Related large scale quantum effects: **superfluidity, quantized vortices**.
- Experimental realization by **Cornell, Wieman** and **Ketterle 1995**.
- 2001 Nobel Prize in Physics.

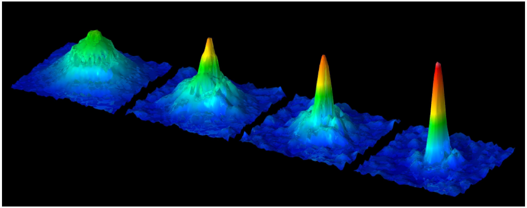


Crash course on BEC

- Consider an **ideal** gas of N bosons at $T > 0$ in a box with p.b.c.
- **Bose-Einstein distribution**:

$$\gamma^{\mathrm{id}}(p) = \frac{1}{\exp((p^2 - \mu_0)/T) - 1}$$

  describes the expected number of particles with momentum p
- on the other hand $\sum_p \gamma^{\mathrm{id}}(p) = N$ (fixed by the chemical potential)
- for every $p \neq 0$ and $\mu_0 < 0$ we have $\gamma^{\mathrm{id}}(p) \to 0$ as $T \to 0$



BEC phase transition

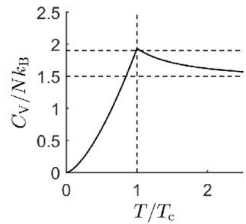
- in the **thermodynamic limit**, i.e. $N, V \to \infty$, $N/V = \rho = const$

$$\rho = \frac{1}{V}\sum_p \gamma^{\mathrm{id}}(p) \xrightarrow[V\to\infty]{} \int \gamma^{\mathrm{id}}(p)dp \leqslant \int \gamma^{\mathrm{id}}(p)\Big|_{\mu_0=0} dp =: \rho_{cr}(T)$$

- thus $\rho \leqslant \rho_{cr}(T) \xrightarrow[T\to 0]{} 0$. What is wrong?
- Below critical temperature **macroscopic occupation** of $p = 0$ mode:

$$\gamma^{\mathrm{id}}(0) = \frac{1}{\exp(-\beta\mu_0) - 1} \sim O(N)$$

- Second order **phase transition**



Some abstract nonsense...

- What about **interacting** systems?
- in **quantum statistical mechanics** the equilibrium is described by **Gibbs state**

$$G_N = \frac{e^{-\beta\mathcal{H}_N}}{\mathrm{Tr}_{\mathfrak{h}^N}\, e^{-\beta\mathcal{H}_N}}$$

- $\mathcal{H}_N$ - N-body Hamiltonian, $\mathfrak{h}^N$- N-body Hilbert space
- expectation values

$$\langle A\rangle = \mathrm{Tr}(AG_N)$$

- Bose-Einstein distribution is just $\langle a_p^* a_p\rangle$ for the ideal gas...
- ...which is just the diagonal of the **one-body density matrix**

$$\gamma^{(1)}_{G_N} = \mathrm{Tr}_{2\to N}[G_N]$$

Definition

A sequence of states G_N displays **Bose–Einstein condensation** iff

$$\liminf_{N\to\infty} \sup_{\|\psi\|_2=1} \frac{\langle\psi, \gamma^{(1)}_{G_N}\psi\rangle}{N} > 0$$

Interacting systems

- **Many-body** Hamiltonian (again, box with p.b.c)

$$\mathcal{H}_N = \sum_{i=1}^{N} -\Delta_i + \sum_{1 \leqslant i < j \leqslant N} v(x_i - x_j)$$

Proving BEC in the thermodynamic limit remains open problem!

- even in the ground state!
- simpler models: **mean-field** (MF) scaling: fixed size of box

$$\mathcal{H}_N = \sum_{i=1}^{N} -\Delta_i + \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} v(x_i - x_j)$$

- **BEC proven** in MF: $T = 0$ **Lieb, Seiringer 2002**, $T > 0$ **Deuchert, Seiringer 2020**

Spontaneous Symmetry Breaking (SSB)

There is SSB if a symmetry of the Hamiltonian (or Lagrangian) of a system is not present in the state under consideration (usually a ground state or a thermal equilibrium state).

- Crucial concept in Quantum Field Theory (**Nambu, Goldsotne, Higgs**...) and Statistical Physics (**Anderson, Mermin, Wagner, Hohenberg**, ...).
- **Paradigm** of statistical physics:

  Phase transitions are accompanied by SSB.

- Simple example to keep in mind: magnetization below Curie point.
- Hard to prove! Limited results: **Dyson, Lieb, Simon, Fröhlich, Spencer, Bałaban, ...** (all for lattice models)

SSB in Bose gas

- $U(1)$ symmetry of the Bose gas due to **particle number conservation**.
- second quantized Hamiltonian (momentum basis)

$$\mathcal{H} = \sum_{p \in \Lambda^*} p^2 a_p^* a_p + \frac{1}{2N} \sum_{p,u,v \in \Lambda^*} \hat{v}(p) a_{u+p}^* a_{v-p}^* a_u a_v$$

- invariant under the transformation

$$a_p \mapsto e^{i\tau} a_p, \qquad \forall p$$

- Consequently, its Gibbs state has a definite number of particles.
- **We proof that in the limit $N \to \infty$ mean-field limit this gauge invariance is spontaneously broken**:

$$\langle a_0 \rangle_{q-a} \neq 0$$

Bogoliubov quasi-averages

Bogoliubov 1961 introduces **quasi-averages** which provide a mathematical scheme how to describe SSB in statistical mechanics:

1. couple the Hamiltonian with a symmetry breaking term

$$\mathcal{H}^\lambda = \mathcal{H} + \lambda\sqrt{N}(a_0^* + a_0)$$

2. consider the expectation values in the perturbed Hamiltonian

$$\langle A \rangle_\lambda = \mathrm{Tr}(AG^\lambda)$$

   where

$$G^\lambda = \frac{e^{-\beta(\mathcal{H}^\lambda - \mu_N \mathcal{N})}}{\mathrm{Tr}\, e^{-\beta(\mathcal{H}^\lambda - \mu_N \mathcal{N})}}$$

3. a **quasi-average** of the observable A is then

$$\langle A \rangle_{q-a} := \lim_{\lambda\to 0} \lim_{N\to\infty} \langle A \rangle_\lambda$$

A rigorous result...

- We consider the mean-field Hamiltonian with a repulsive two-body interaction satisfying some regularity conditions.
- We set the chemical potential μ_N such that

$$\mathrm{Tr}[\mathcal{N} G^0] = N$$

- $N_0(\beta, N)$ - number of particles in the $p = 0$ mode in the ideal gas

Theorem (Deuchert, Nam, N. 2024)

Let $\beta = \kappa\beta_{\mathrm{c}}$ with β_{c} the critical temperature of the ideal gas. Let $\kappa \in \mathbb{R}_+$. Then we have

$$\lim_{\lambda \to 0} \lim_{N \to \infty} \frac{|\,\mathrm{Tr}[a_0 G^\lambda]|}{N^{1/2}} = \lim_{N \to \infty} \sqrt{\frac{N_0(\beta, N)}{N}} = \sqrt{\left[1 - \frac{1}{\kappa^{3/2}}\right]_+}.$$

Relation between SSB and BEC

- as mentioned earlier **proof of BEC in the thermodynamic limit remains open**;
- quasi-averages scheme the same, but limit $V \to \infty$;
- **Lieb, Seiringer, Yngvason 2005, Süto 2005** proved that

$$\text{BEC}^{TL} \implies (\text{BEC})^{TL}_{q-a} \iff SSB^{TL}$$

- for the mean-field model we prove **full equivalence**

Theorem (Deuchert, Nam, N. 2024)

$$\lim_{N\to\infty} \frac{\operatorname{Tr}[a_0^* a_0 G^0]}{N} = \left[1 - \frac{1}{\kappa^{3/2}}\right]_+ = \lim_{\lambda\to 0} \lim_{N\to\infty} \frac{|\operatorname{Tr}[a_0 G^\lambda]|}{N^{1/2}}$$

$$\text{BEC}^{MF} \iff (\text{BEC})^{MF}_{q-a} \iff SSB^{MF}$$

Resume

Conclusions:

- BEC phase transition is accompanied by $U(1)$ symmetry breaking;
- we prove this fact for the mean-field Bose gas;
- we prove BEC and SSB are equivalent.

Outlook

- superfluidity, i.e. $\langle a_p a_{-p} \rangle$;
- relation between superfluidity and BEC;
- thermodynamic limit :)

Thank you for your attention!