# Matrix model on the crossroad of integrability and representation theory

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## Correlation functions in Gaussian matrix model

The partition function for the $\beta = \frac{1}{\alpha}$-deformed Gaussian eigenvalue model can be written in eigenvalue form

$$Z \stackrel{\text{def}}{=} \int \left( \prod_{i=1}^{N} dx_i \right) w_\beta(x) \exp\left[ -\sum_{i=1}^{N} \frac{x_i^2}{2} \right]. \tag{1.1}$$

where $w_\beta(x) = \prod_{1 \le i < j \le N} (x_i - x_j)^{2\beta}$ is the Vandermonde and $x_i$ are diagonalized elements of Hermitian matrix $H = h_{ij}$. Our goal is to find a multipoint correlation of the gaussian model

$$\mathbb{E}\left[ Tr(H^{k_1}) \dots Tr(H^{k_l}) \right] = \mathbb{E}\left[ \left( \sum_{i=1}^{N} x_i^{k_1} \right) \dots \left( \sum_{i=1}^{N} x_i^{k_l} \right) \right] \tag{1.2}$$

We add generating function with parameter $q_k$ to the partition function:

$$Z(q_k) = \int \left( \prod_{i=1}^{N} dx_i \right) w_\beta(x) \exp\left[ -\sum_{i=1}^{N} \frac{x_i^2}{2} + x_i \right] \exp\left[ \beta \sum_{k=1}^{\infty} \frac{q_k}{k} \sum_{i=1}^{N} x_i^k \right]. \tag{1.3}$$

# How to attack the problem

# Symmetric function:

There are many symmetric polynomials.

Monomial symmetric functions:

$$\mathbf{m}_\lambda = \sum_{\mu \sim \lambda} x^\mu, \tag{2.4}$$

where $\mu \sim \lambda$ means rearrangement of parts of $\lambda$.

Powersum symmetric functions:

$$p_\lambda = \prod_{i \in \ell(\lambda)} p_{\lambda_i} \quad \text{and,} \quad p_k = \sum_{i=1}^{n} x_i^k, \tag{2.5}$$

where $\ell(\lambda)$ is the length of partition $\lambda$ and $\lambda_i$ its $i$-part. We equipped our ring with an inner product $\langle \bullet, \bullet \rangle$

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}, \qquad z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!, \tag{2.6}$$

$m_i$ is the number the parts in *lambda* equals to $i$.

# Deformation of Hall inner product and Jack polynomials:

We can deform the Hall inner product

$$\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda,\mu} \alpha^{\ell(\lambda)} z_\lambda, \tag{2.7}$$

with $\alpha = \frac{1}{\beta} \in \mathbb{R}$. This deformation allow us to have a deformation of Schur functions with respect to this new inner product.

Jack polynomials defines uniquely by the fillowing conditions:

$$\langle P_\lambda, P_\mu \rangle_\alpha = 0, \text{ if } \lambda \neq \mu,$$

$$P_\lambda = \sum_{\mu \leq_d \lambda} C_{\lambda\mu} \mathbf{m}_\mu$$

$$[\mathbf{m}_\lambda] P_\lambda = 1, \quad \text{P-normalization.}$$

$$[\mathbf{m}_\lambda] J_\lambda = |\lambda|!, \quad \text{J-normalization.}$$

# Dunkl Operator

# Dunkl Operator

Dunkl operator defines on using a certain data:
**Definition:**

- Let $R$ be a root system.
- Let $G$ be a reflection group on $R^\vee$
- $k : R \to \mathbb{C}$ a $G$-invariant function.
- $\sigma_\alpha X := X - \frac{2\langle X,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ is a reflection aloung the root $\alpha \in R$.

Then the Dunkl operator for $\xi \in \mathbb{R}^N$ is

$$T_\xi f(x) := \partial_\xi + \sum_{\alpha\in R_+} k_\alpha \langle\alpha,\xi\rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle\alpha, x\rangle}. \tag{3.8}$$

We set $T_{\xi_i} = T_i$ for $\xi_i \in \mathbb{R}^N$.

# Dunkl Kernel and Bessel function

**Definition/Theorem** Dunkl Kernel is $f = E_k(\bullet, y)$ such that

$$T_\xi f = \langle \xi, y \rangle f, \qquad f(0) = 1. \tag{3.9}$$

It is a unique and real analytic solution for $Re(k > 0)$ [Dunkl].

Some properties:

- $E_k(y, x) = E_k(x, y)$.
- $E_k(gx, gy) = E(x, y)$, for $g \in G$.

And the convolution theorem [Dunkl]

$$\int_{\mathbb{R}^N} E_k(x, y) E_k(x, z) e^{-|x|^2/2} w_k(x) dx = c_k e^{(y,y)/2+(z,z)/2} E_k(z, y).$$

# Generalized Bessel function and spherical part of algebra

The following two theorems by [Opdam] and [Okounkov] connect the theory of Dunkl operators to matrix models.

**Definition:** Generalized bessel function is defined as

$$F_k(x, y) := \frac{1}{|G|} \sum_{f \in G} E_k(gx, y). \tag{3.10}$$

**Theorem:** [Okounkov]

$$F_{\frac{1}{\alpha}}(x, y) = \sum_{\lambda} \frac{P_\lambda(x; \alpha) P_\lambda(y; \alpha)}{(n/\alpha)_\lambda \mathfrak{p}_\lambda}. \tag{3.11}$$

where $(u)_\lambda = \prod_{(i,j) \in \lambda} (u + (j-1) - (i-1)/\alpha)$, and $\mathfrak{p}_\lambda = (P_\lambda, P_\lambda)$ and $Re(\alpha) > 0$.

# Result:

Using Okounkov Bessel function formula and Dunkl convolution theorem we can evaluate the average of two Jack polynomials

**Theorem [P.K. P. Sułkowski: arXiv:24XX.XXXX]:**

$$\mathbb{E}[P_\mu(x;\alpha)P_\beta(x;\alpha)] = \alpha^{-|\mu|-|\beta|}J_\mu(1;\alpha)J_\beta(1;\alpha)\times \left\langle P_\mu(y;\alpha), \left\langle P_\beta(z;\alpha), e^{|y|^2/2+|z|^2/2}\sum_\lambda \frac{P_\lambda(y;\alpha)P_\lambda(z;\alpha)}{(n/\alpha)_\lambda \mathfrak{p}_\lambda}\right\rangle\right\rangle \tag{3.12}$$

**Corollary:**

$$\mathbb{E}[P_\mu(x;\alpha)e^{-p_1(x)}] = \alpha^{-|\mu|}J_\mu(1;\alpha)e^{\frac{-N}{2}}\left[p_2^{|\mu|/2} - p_1^{|\mu|}\right]P_\mu \tag{3.13}$$

**Thanks.**

# Cauchy identity

Cauchy identity for Jack polynomials is

$$\exp\left[\beta\sum_{k=1}^{\infty}\frac{p_k\bar{p}_k}{k}\right]=\sum_{\lambda}\frac{P_\lambda\{p_k\}P_\lambda\{\bar{p}_k\}}{\langle P_\lambda,P_\lambda\rangle}. \tag{3.14}$$

here $p_k$ and $\bar{p}_k$ are power sum polynomials of possibly different sets if variables. Putting $\bar{p}_k = a_2\beta^{-1}\delta_{k,2}$ we see that

$$\exp\left[\frac{a_2 p_2}{2}\right]=\sum_{\lambda}\frac{P_\lambda\cdot P_\lambda\{a_2\beta^{-1}\delta_{k,2}\}}{\langle P_\lambda,P_\lambda\rangle}. \tag{3.15}$$

But the left hand side can also obtain by mutiplying $p_2$ n-times on $P_\varnothing$. This gives us the expression for C's

$$\sum_{\lambda^{(2)}\dots\lambda^{(2n-2)};\lambda^{(2n)}=\lambda} C_{\phi\lambda^{(2)}}C_{\lambda^{(2)}\lambda^{(4)}}\dots C_{\lambda^{(2n-2)}\lambda^{(2n)}}=\frac{2^{|\lambda|/2}(|\lambda|/2)!}{a_2^{|\lambda|/2}}\frac{P_\lambda\{a_2\beta^{-1}\delta_{k,2}\}}{\langle P_\lambda,P_\lambda\rangle}. \tag{3.16}$$

# A and Peri rule

First, we notice that

$$A_{\mu\lambda} = \frac{\langle P_\lambda, p_1^2 P_\mu \rangle}{\langle P_\lambda, p_2 P_\mu \rangle} = \frac{1}{1 - 2\frac{\langle P_\lambda, P_{(1,1)} P_\mu \rangle}{\langle P_\lambda, P_1^2 P_\mu \rangle}}. \tag{3.17}$$

The last equality follows from $P_{\{1\}} = p_1$ and $P_{\{1,1\}} = \frac{1}{2}\left(p_1^2 - p_2\right)$. Jack polynomials are satisfying a rule known as Pieri rule

$$P_{(1^r)} P_\mu = \sum_\lambda c^\lambda_{\mu,(1^r)} P_\lambda, \tag{3.18}$$

where $\lambda - \mu$ is vertical r-strip and $c^\lambda_{\mu,(1^r)}$ is known object. Using this rule we can calculate $A_{\mu\nu}$

$$\beta(1-\beta)A_{\mu\lambda} = (j_2 - j_1 + \beta(i_1 - i_2))^2 - (1 - \beta + \beta^2) \tag{3.19}$$

# Peri rule coefficients

The coefficients in Peri rule 3.18 is given by

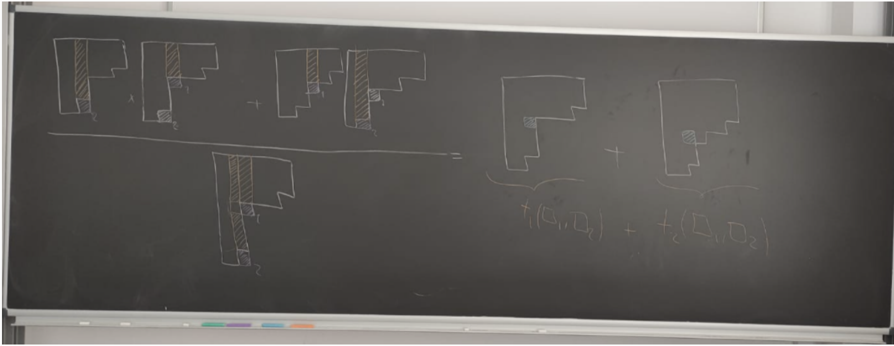
$$c^{\lambda}_{\mu,(1^r)} = \prod_{s\in X(\lambda/\mu)} \frac{h^{\lambda}_{*}(s)h^{*}_{\mu}(s)}{h^{\mu}_{*}(s)h^{*}_{\lambda}(s)} \tag{3.20}$$

$h^{*}_{\lambda}(s)$ and $h^{\lambda}_{*}(s)$ are respectively the upper and lower hook lengths of the box $s$:

$$\begin{aligned} h^{*}_{\lambda}(s) &= \beta^{-1}(a_{\lambda}(s)+1)+l_{\lambda}(s) \\ h^{\lambda}_{*}(s) &= \beta^{-1}a_{\lambda}(s)+l_{\lambda}(s)+1 \end{aligned} \tag{3.21}$$

$$\frac{\langle P_{\lambda}, P_1^2 P_{\mu}\rangle}{\langle P_{\lambda}, P_{(1,1)}P_{\mu}\rangle} = \frac{\sum_{\sigma=\mu+\Box} c^{\lambda}_{\sigma,1}c^{\sigma}_{\mu,1}}{c^{\lambda}_{\mu,(1,1)}} = \frac{c^{\mu+\Box_1+\Box_2}_{\mu+\Box_2,1}c^{\mu+\Box_2}_{\mu,1} + c^{\mu+\Box_1+\Box_2}_{\mu+\Box_1,1}c^{\mu+\Box_1}_{\mu,1}}{c^{\mu+\Box_1+\Box_2}_{\mu,(1,1)}} \tag{3.22}$$

# Depicted version of Peri rule



# References


[1] L. Lando, A. Zvonkin , Graphs on Surfaces and Their Applications

[2] L. Cassia, R. Lodin and M. Zabzine, JHEP **10** (2020), 126 doi:10.1007/JHEP10(2020)126 [arXiv:2007.10354 [hep-th]].

[3] C. Itzykson and J.-B. Zuber. Matrix integration and combinatorics of modular groups. Communications in Mathematical Physics, 134(1):197 – 207, 1990. doi: cmp/1104201618. URL https://doi.org/.

# Jack polynomials

Jack Polynomials is beta deform extension of Schur polynomials. We can define Jack polynomial P similar to Schur polynomial by adding an extra weight.

$$P_\lambda = \sum_T \psi_T(\beta) \prod_{s\in\lambda} z_{T(s)}, \tag{4.23}$$

where extra weight $\psi_T$ is given with respect to sequence of partition in Young diagram, $\emptyset = \nu_1 \to \nu_2 \to \cdots \to \nu_n = \lambda$.

$$\psi_T(\beta) = \prod_i \psi_{\nu_{i+1}/\nu_i} \qquad \text{where,} \tag{4.24}$$

$$\psi_{\lambda/\mu} = \prod_{s\in R_{\lambda/\mu} - C_{\lambda/\mu}} \frac{arm_\mu(s) + \beta(leg_\mu(s)+1)}{arm_\mu(s) + \beta leg_\mu + 1} \frac{arm_\lambda(s) + \beta(leg_\lambda(s)+1)}{arm_\lambda(s) + \beta leg_\lambda + 1} \tag{4.25}$$

where arm(s) is number of boxes in the right of s and leg(s) is number of boxes below s. Jack polynomials form an orthogonal basis $\langle P_\lambda, P_\mu \rangle = 0$ whenever $\lambda \neq \mu$.

If we put $\beta = 1$,we get $\psi = 1$, and we recover definition of Schur.