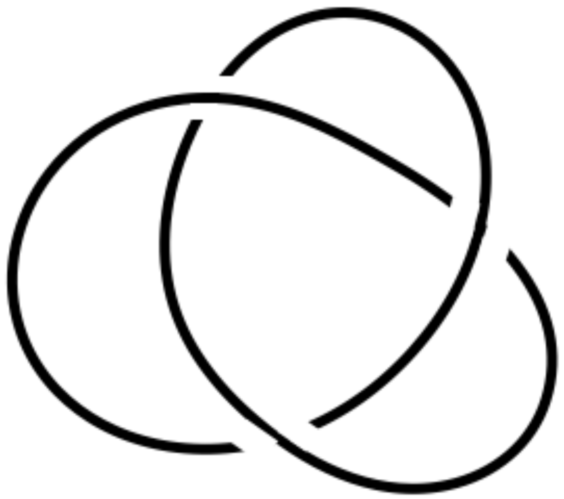
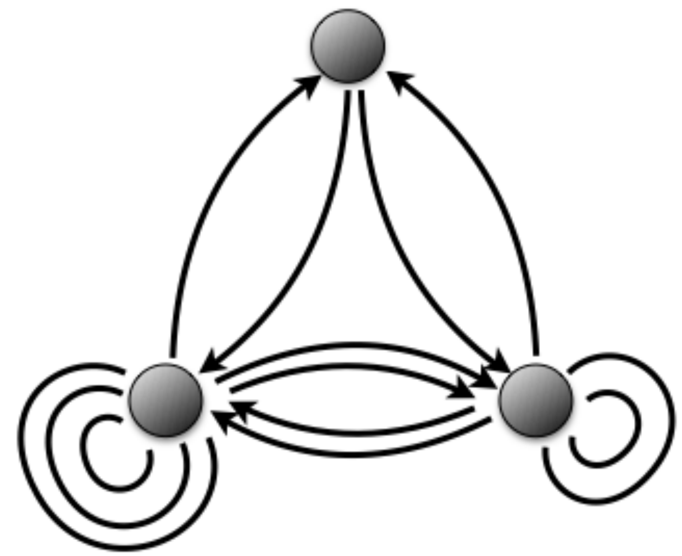
# Quantum fields, strings, knots, quivers, and physical mathematics






Piotr Sułkowski

Chair of Quantum Mathematical Physics
Faculty of Physics, University of Warsaw


Theoretical Physics Symposium, December 2024

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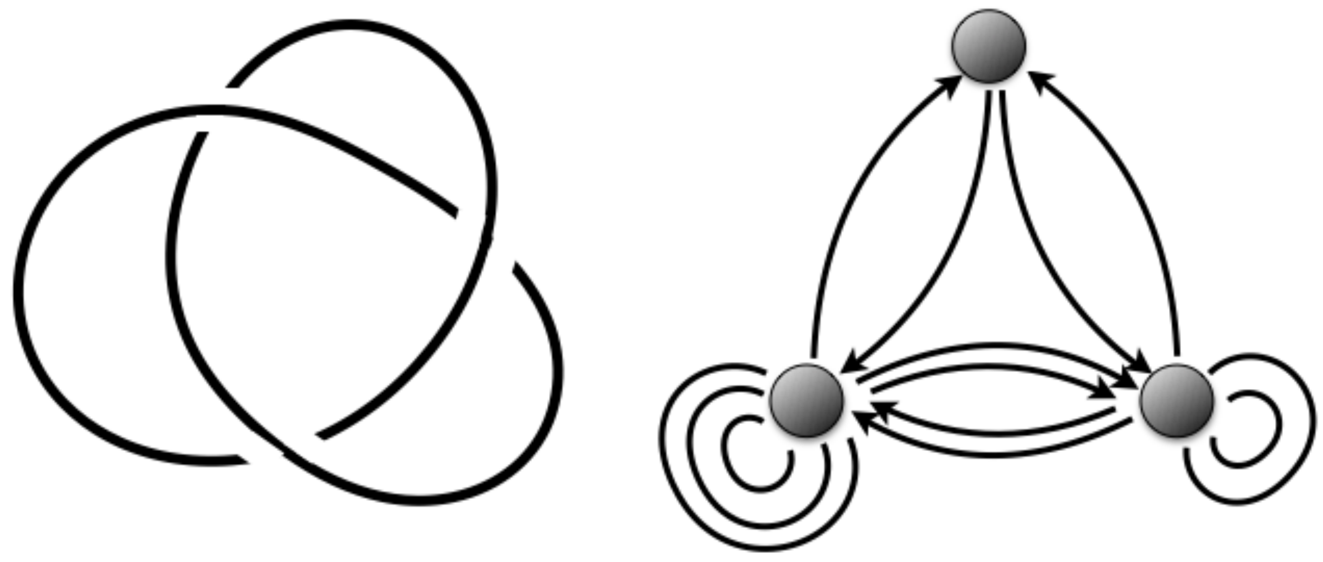


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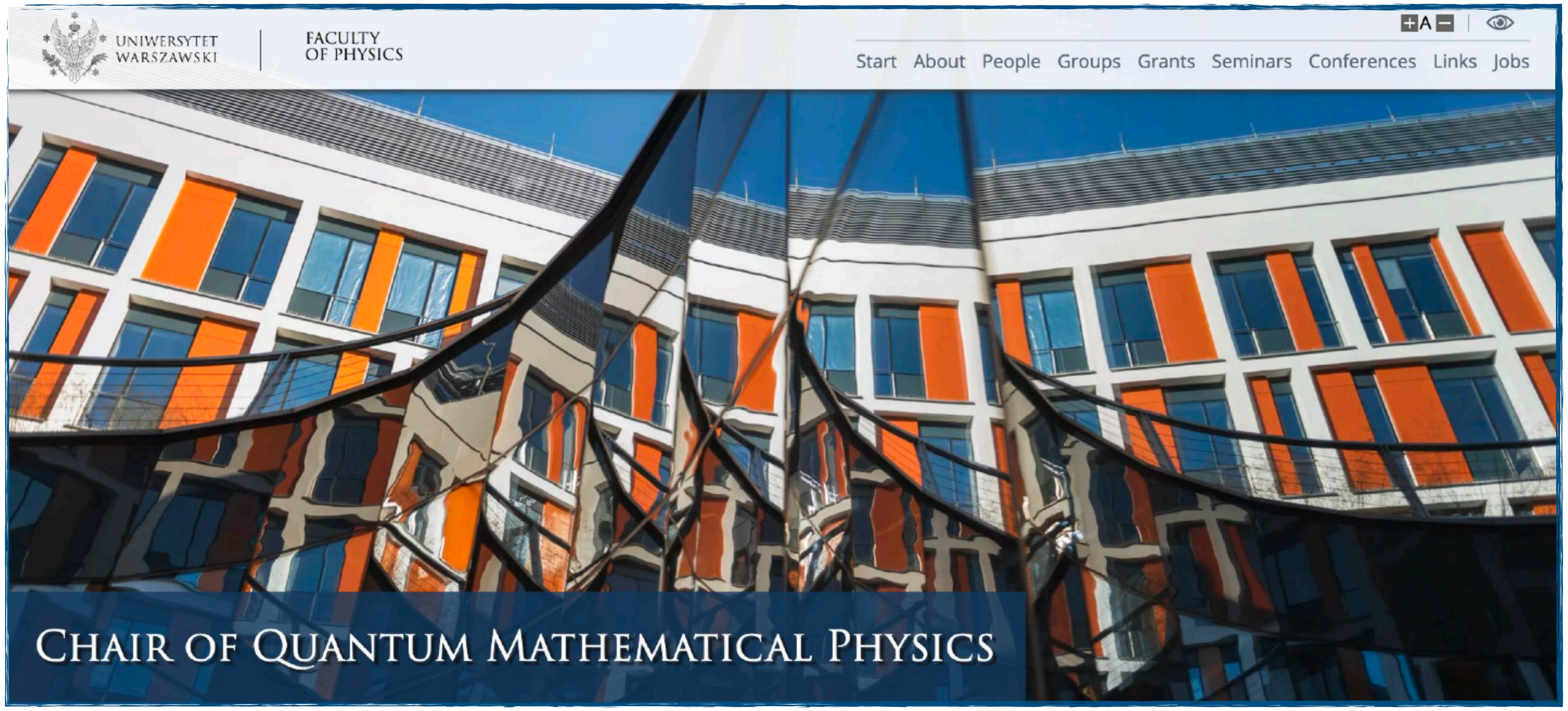


- Quantum Fields and Strings
- Strongly Correlated Systems and Quantum Integrability
- Celestial Holography
- Quantum Information in Quantum Gravity

*Faculty members and research groups:*

Paweł Caputa, Miłosz Panfil, Jacek Pawełczyk, P.S., Tomasz Taylor

    

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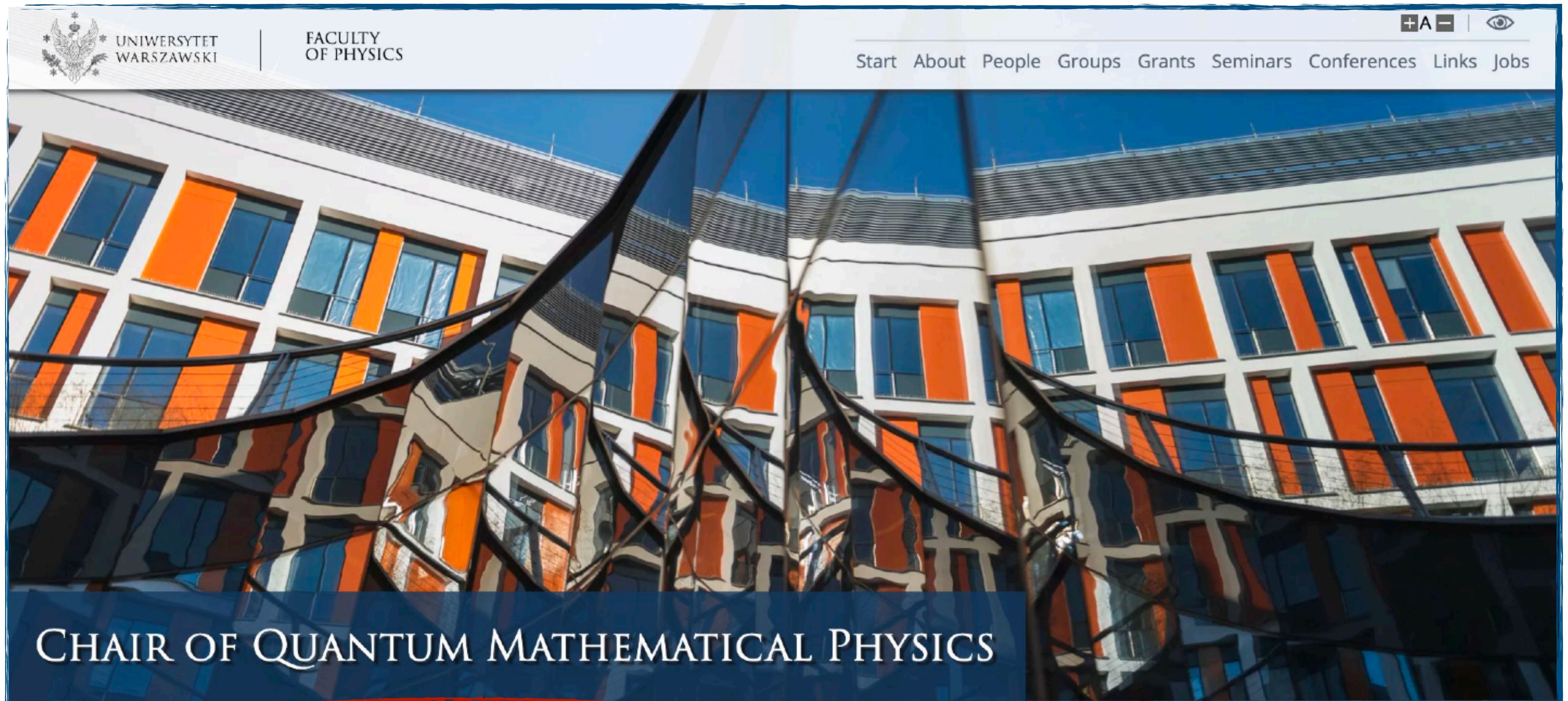


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*Our group:* **D.Bryan, L.Guerrini, P.Karimi, S.Kalinowski, J.Kenig, H.Potrykus, S.Purkayastha, A.Saha, P.S., ……..**

     

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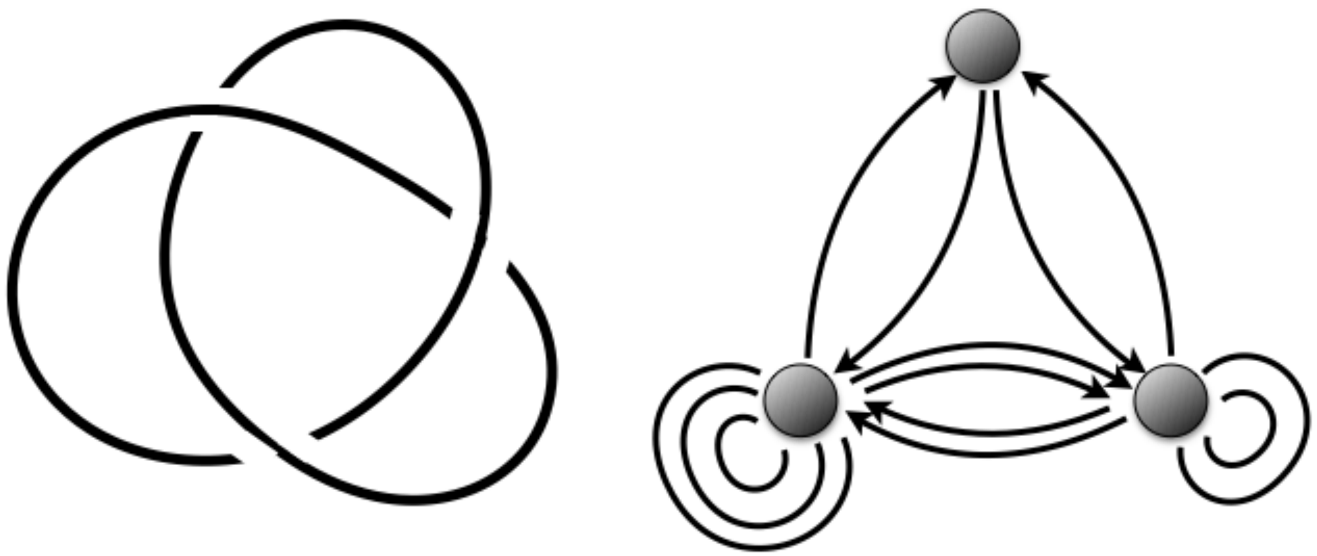

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# Physical mathematics

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From Wikipedia, the free encyclopedia

The subject of **physical mathematics** is concerned with mathematics that is motivated by physics and is considered by some as a subfield of mathematical physics.

The use of the term "Physical Mathematics" is not meant to detract from the venerable subject of Mathematical Physics but rather to delineate a smaller subfield characterized by questions and goals that are often motivated, on the physics side, by quantum gravity, string theory, and supersymmetry, (and more recently by the notion of topological phases in condensed matter physics), and, on the mathematics side, often involve deep relations to infinite-dimensional Lie algebras (and groups), topology, geometry, and even analytic number theory, in addition to the more traditional relations of physics to algebra, group theory, and analysis.

To repeat, one of the guiding principles is the goal of understanding the ultimate foundations of physics. Following the lessons of history, we may reasonably expect this to lead to important new insights in mathematics. But – and here is the central point of this essay – it is also true that getting there is more than half the fun: If a physical insight leads to a significant new result in mathematics, that is considered a success. It is a success just as profound and notable as an experimental confirmation from a laboratory of a theoretical prediction. For example, the discovery of a new and powerful invariant of four-dimensional manifolds is a vindication just as satisfying as the discovery of a new particle.

*Gregory Moore, "Physical Mathematics and the Future" (2014)*

Snowmass Whitepaper: Physical Mathematics 2021

Ibrahima Bah[1], Daniel Freed [2], Gregory W. Moore[3],
Nikita Nekrasov[4], Shlomo S. Razamat[5], Sakura Schäfer-Nameki[6]

March 11, 2022

*arXiv: 2203.05078*

A Panorama Of Physical Mathematics c. 2022

Ibrahima Bah[1], Daniel S. Freed [2], Gregory W. Moore[3],
Nikita Nekrasov[4], Shlomo S. Razamat[5], Sakura Schäfer-Nameki[6]

May 13, 2024

*arXiv: 2211.04467*

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- 2.2 Topological Quantum Field Theory
- 2.3 Algebraic Structures In Quantum Field Theory
  - 2.3.1 Algebraic Structures Related To Operator Product Algebras
  - 2.3.2 Algebraic Structures Associated With Supersymmetric Sectors And BPS States
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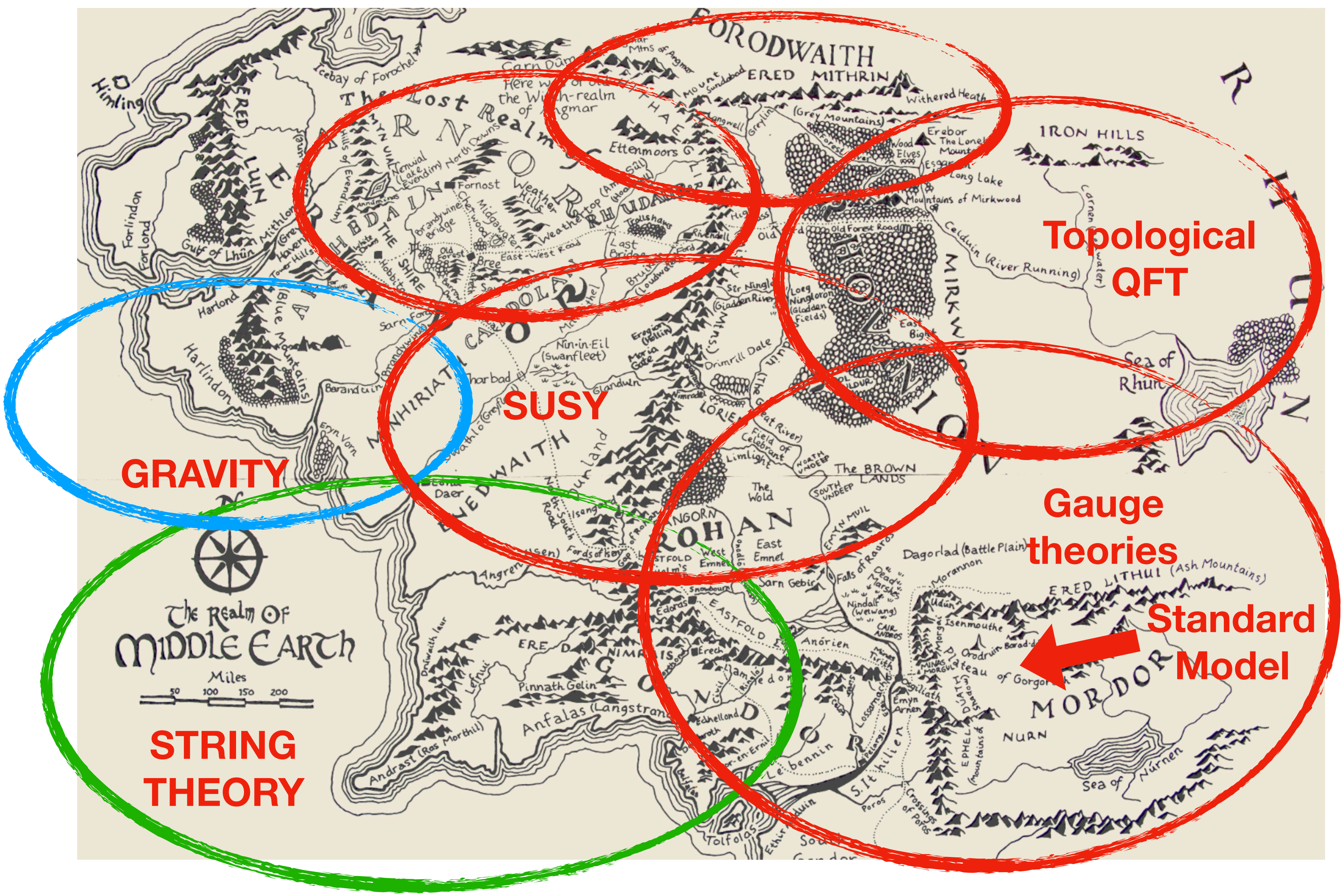
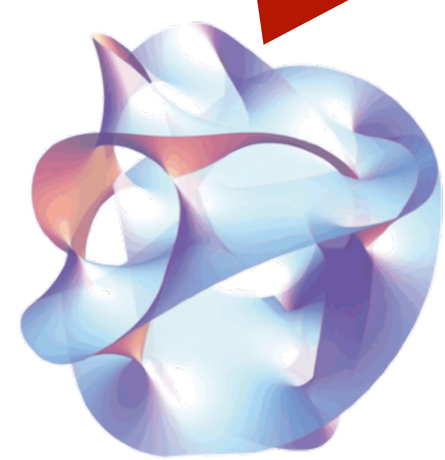
# Space of quantum field theories

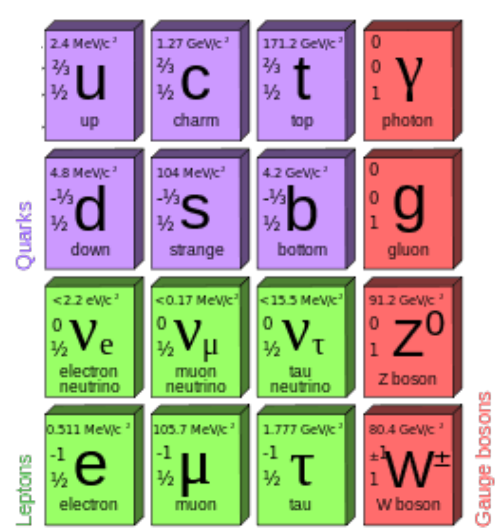


# Gauge theories from string theory

Superstring theory – effective theory in 4 dimensions follows from compactification of 10-dim string theory on a Calabi-Yau manifold.

**10-dim = $R^4$ x** 

*(Beyond) Standard Model theory*

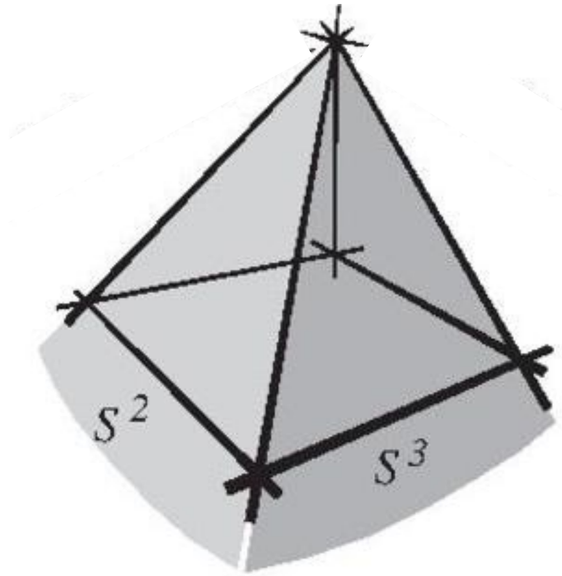



**+ ?**

# (Simple) Calabi-Yau threefold

(Singular) conifold

$$x^2 + y^2 + u^2 + v^2 = 0$$



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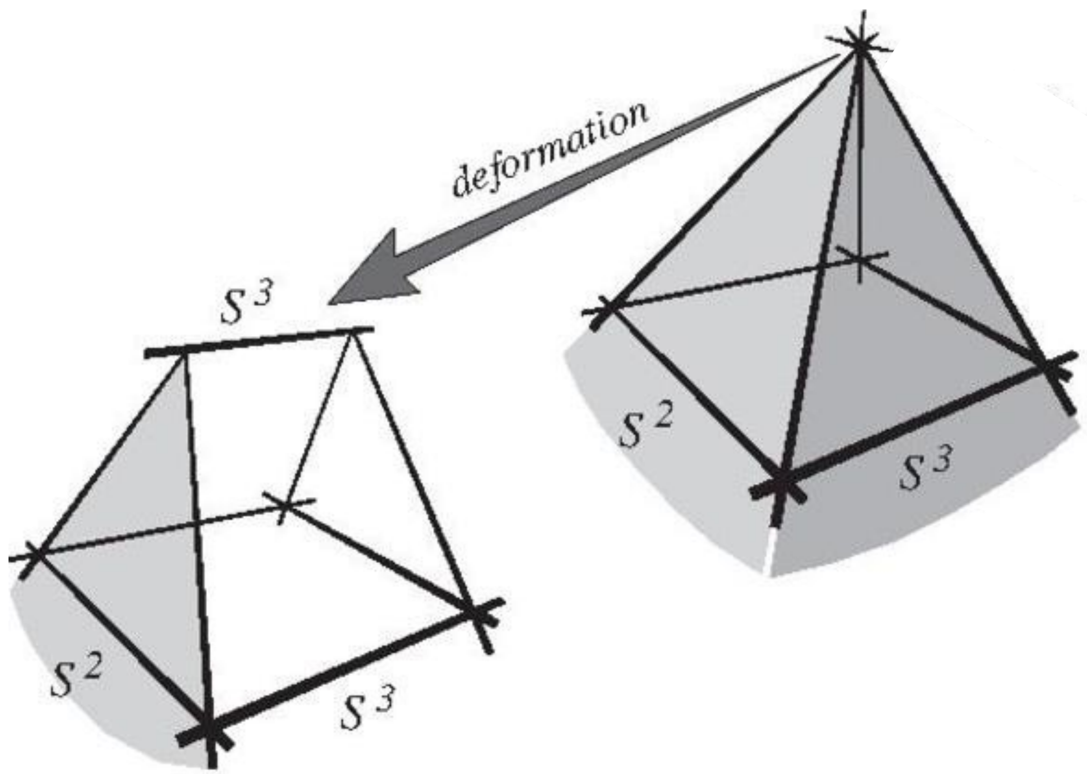


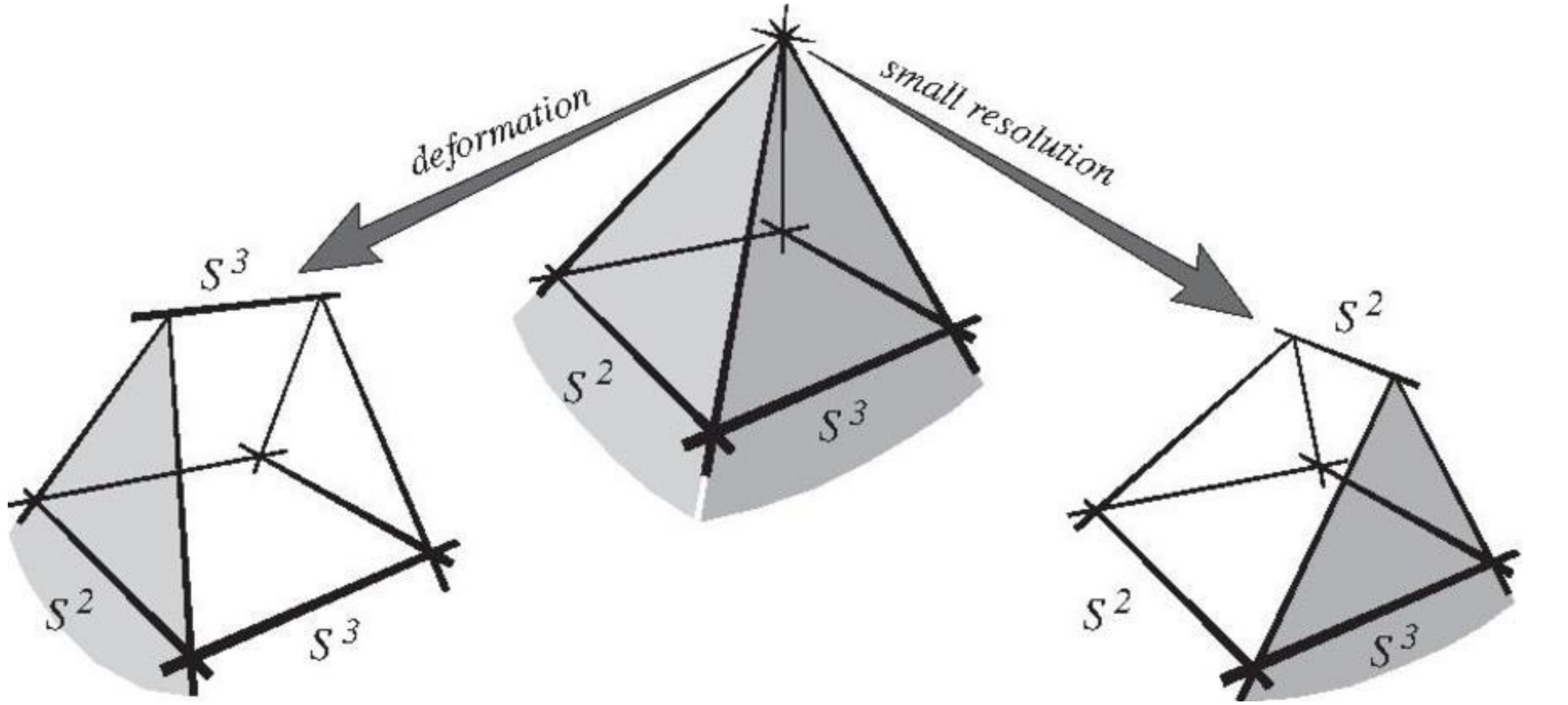

Deformed conifold $T^*S^3$

$$x^2 + y^2 + u^2 + v^2 = t$$

# Geometric transition

(Singular) conifold

$$x^2 + y^2 + u^2 + v^2 = 0$$




Deformed conifold $T^*S^3$

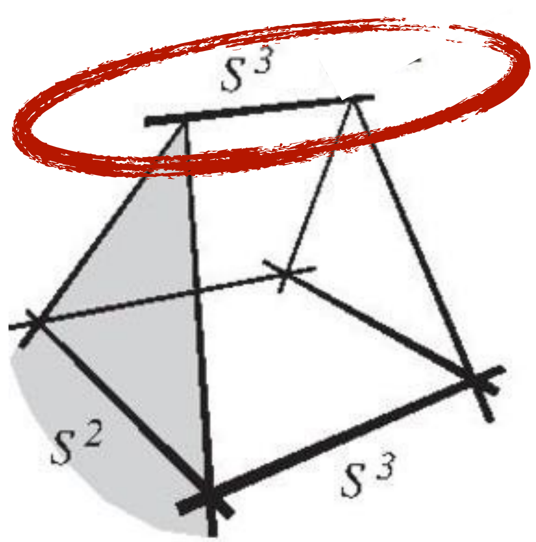
$$x^2 + y^2 + u^2 + v^2 = t$$

Resolved conifold

# Quantum fields from strings

Topological string theory on deformed conifold reduces to Chern-Simons topological field theory on $S^3$, with $SU(N)$ gauge group:

$$S = \frac{k}{4\pi}\int \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

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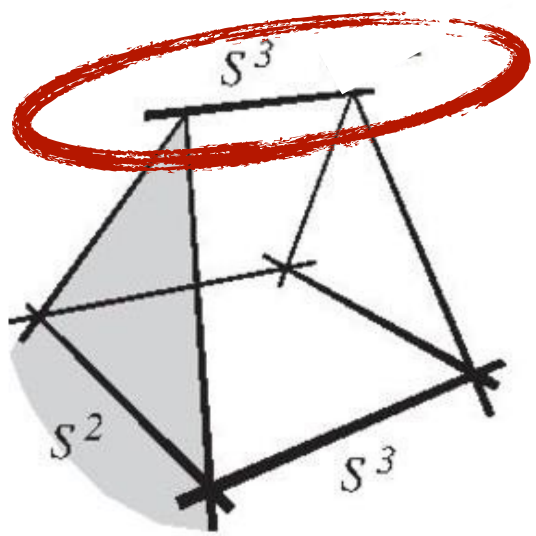


Observables – expectation values of Wilson loops along a closed curve $K$:

$$P(a,q) = \int \mathcal{D}A \left(\mathrm{Tr}\, e^{\oint_K A}\right) e^{iS}$$

$$q = e^{\frac{2\pi i}{k+N}}, \quad a = q^N$$

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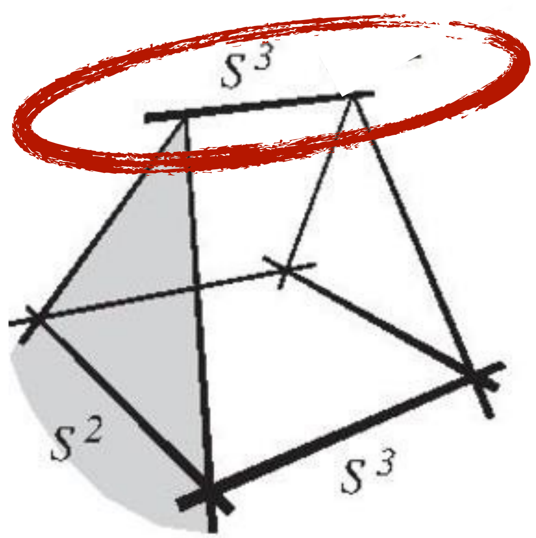


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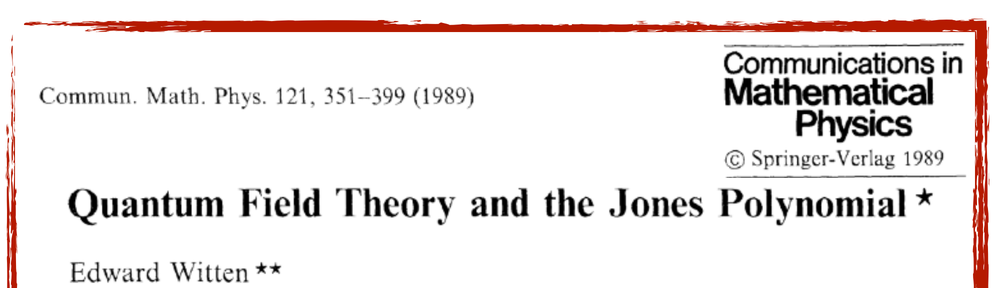
$P(a,q)$ = knot (HOMFLY) polynomials


Commun. Math. Phys. 121, 351–399 (1989)

Communications in Mathematical Physics

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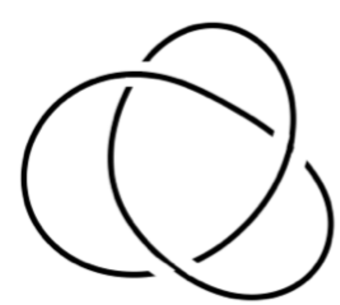
**Quantum Field Theory and the Jones Polynomial** *

Edward Witten **

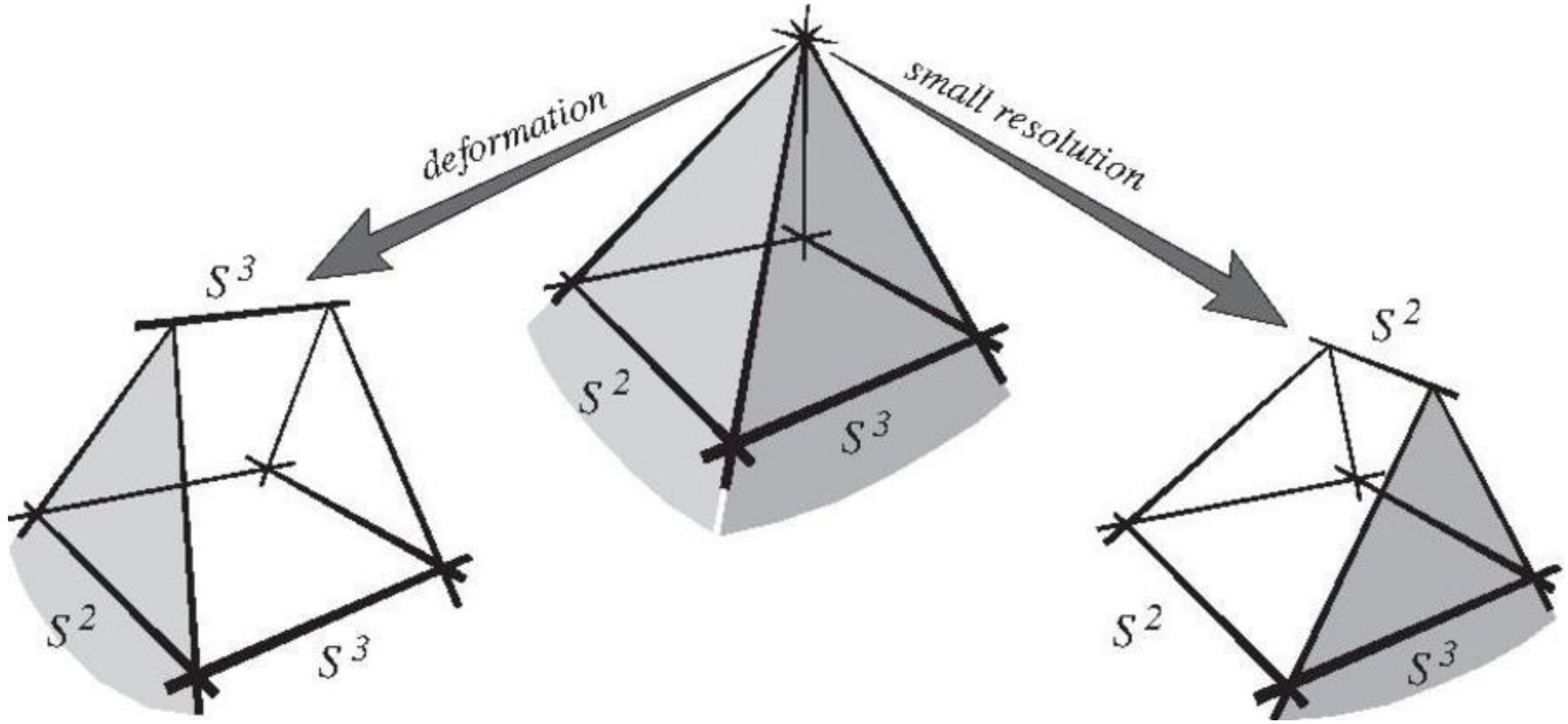


$$P(a,q) = aq^{-1} + aq - a^2$$

# After geometric transition…

(Singular) conifold

$$x^2 + y^2 + u^2 + v^2 = 0$$



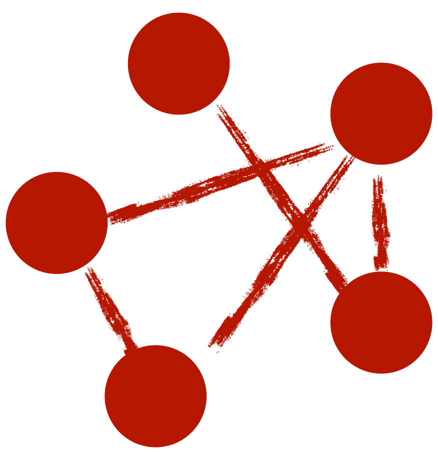

Deformed conifold $T^*S^3$

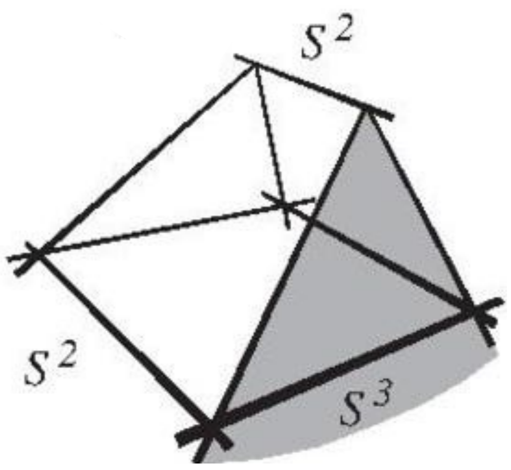
Resolved conifold

# After geometric transition...

After embedding topological string theory in superstrings, in remaining (spacetime) dimensions it reduces to quantum field theory with extended supersymmetry. String amplitudes encode properties of BPS states in this supersymmetric theory:

BPS states



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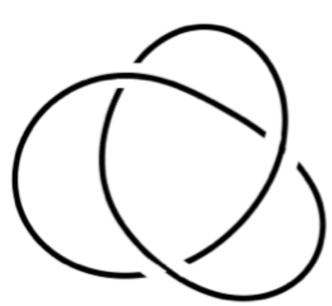
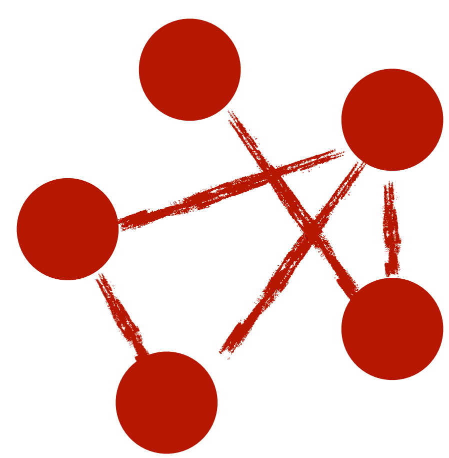
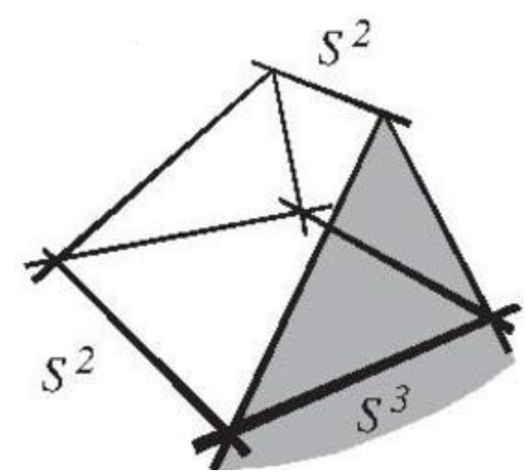







Resolved conifold

Colored HOMFLY polynomials

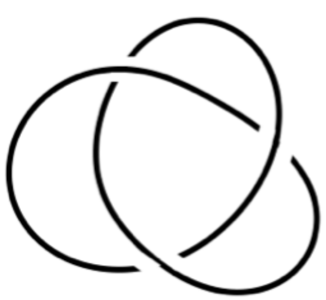
**Knot theory**

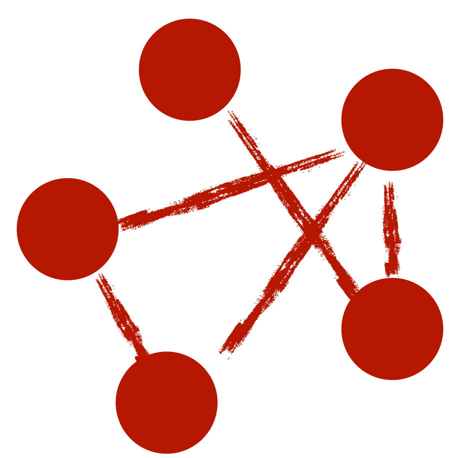
# Ooguri-Vafa (LMOV) invariants

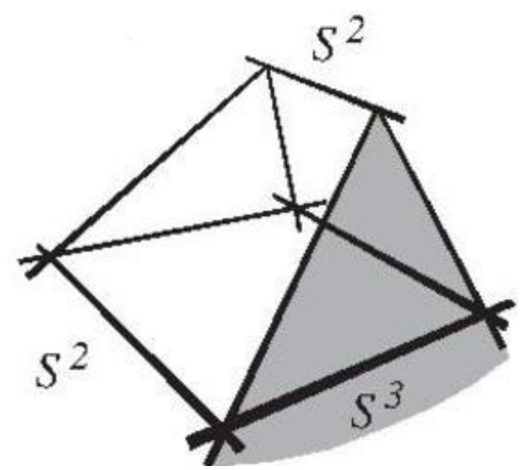
Colored HOMFLY polynomials encode enumeration of these BPS states:

$$\sum_{r=0}^{\infty} x^r P_r(a,q) = \prod_{r,i,j,k} \left(1 - x^r a^i q^{j+2k+1}\right)^{OV_{r,i,j}}$$

BPS states







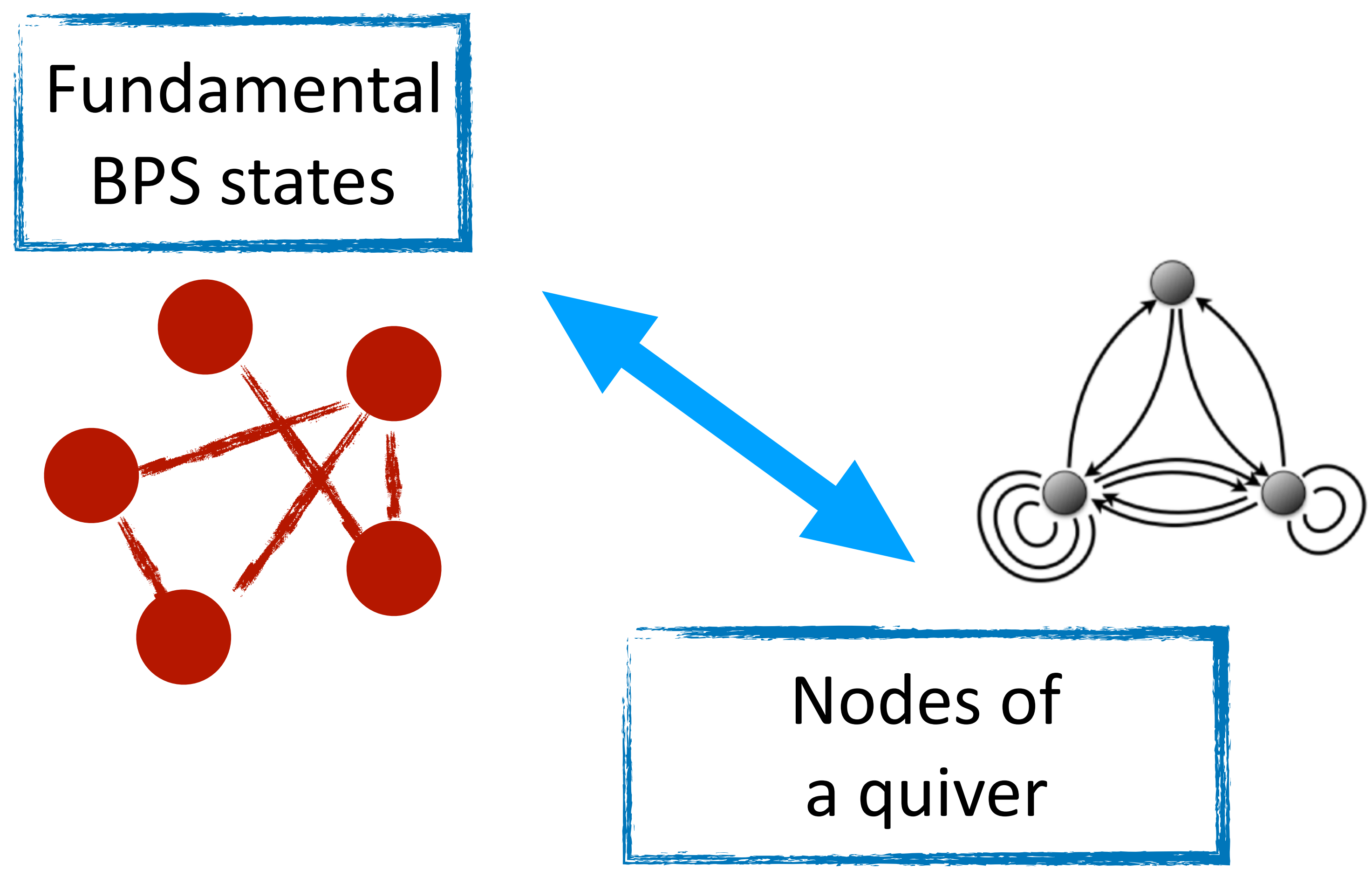

Resolved conifold

Colored HOMFLY polynomials

**Knot theory**

# BPS states and quivers

BPS states in question are in fact bound states of certain fundamental states, described by quiver representation theory:




**Quiver representation theory**

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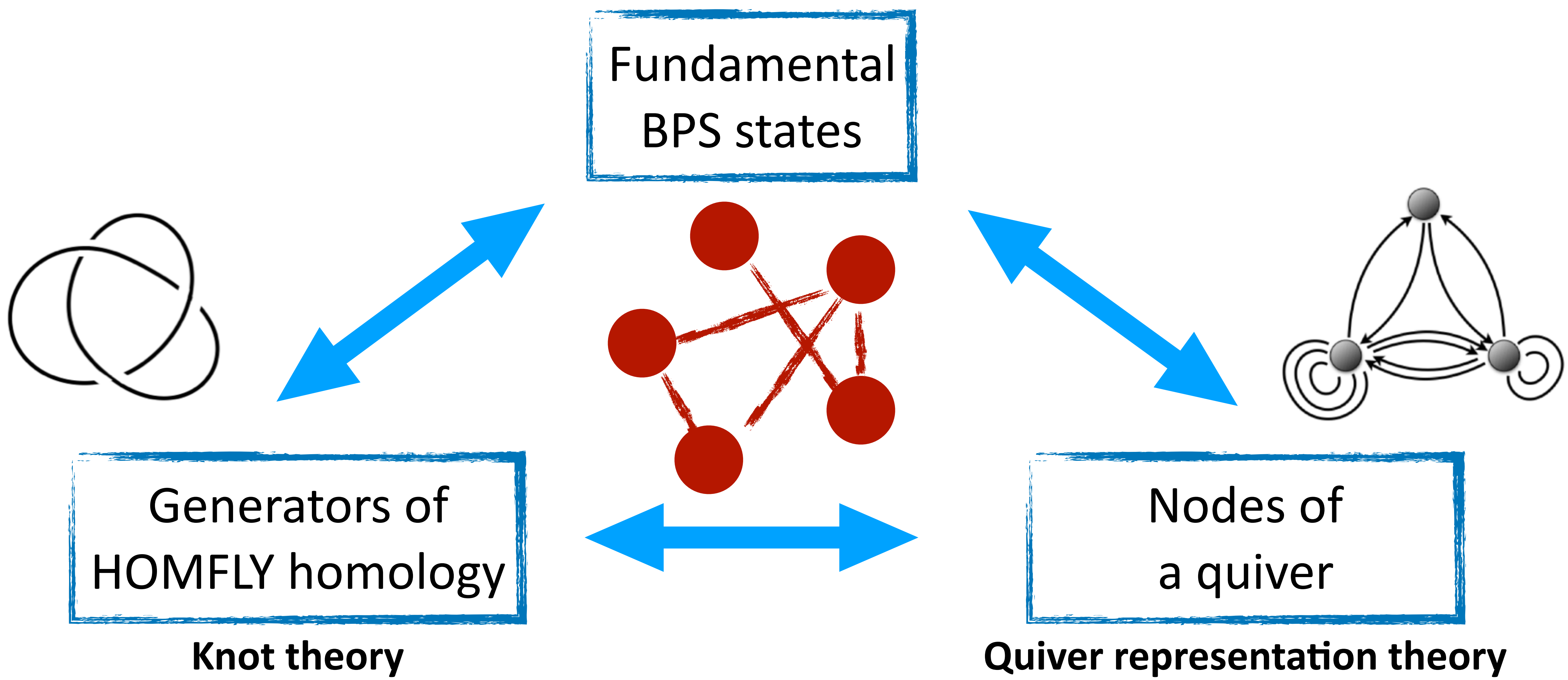
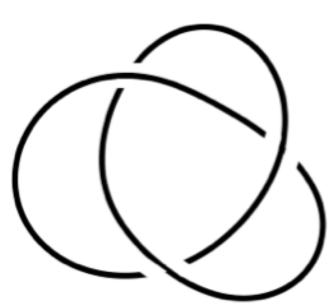


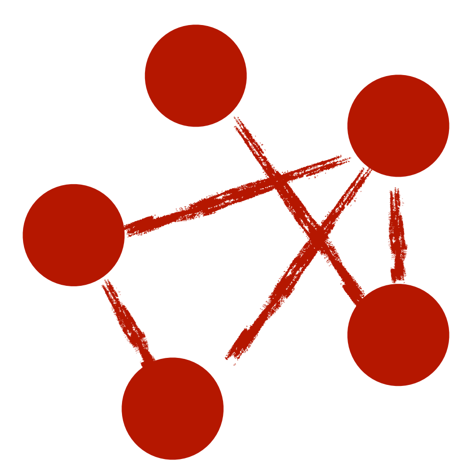
# Knots-quivers correspondence

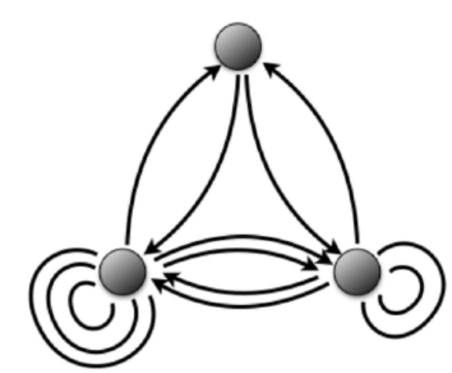
ADV. THEOR. MATH. PHYS.
Volume 23, Number 7, 1849–1902, 2019

**Knots-quivers correspondence**

Piotr Kucharski, Markus Reineke,
Marko Stošić, and Piotr Sułkowski







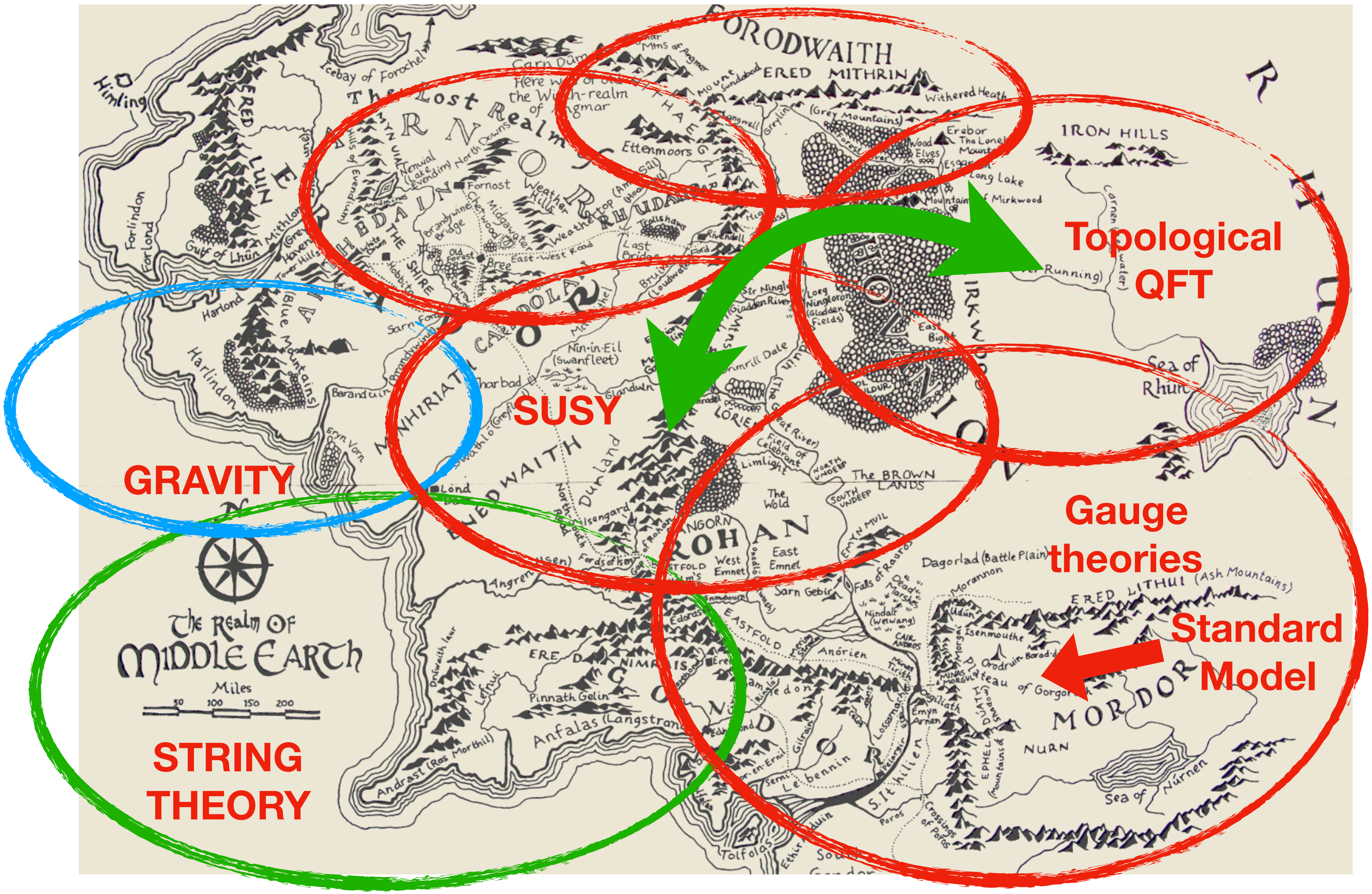
Generators of
HOMFLY homology

**Knot theory**

Nodes of
a quiver

**Quiver representation theory**

# Space of quantum field theories



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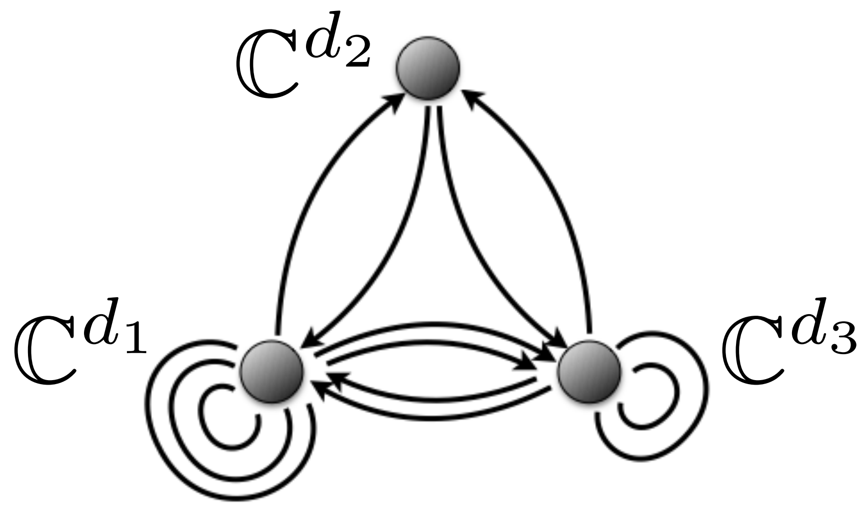
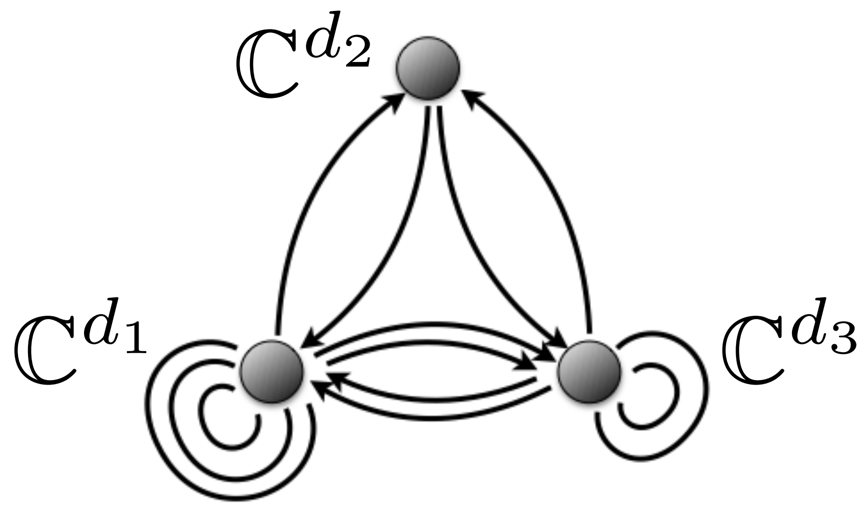



Consider moduli space of maps $\mathbb{C}^{d_i} \to \mathbb{C}^{d_j}$. It is characterized

by motivic Donaldson-Thomas invariants: $\Omega_{d_1,\ldots,d_m;j} \in \mathbb{N}$

$$P_C(x_1,\ldots,x_m) = \sum_{d_1,\ldots,d_m} \frac{(-q)^{\sum_{i,j=1}^m C_{i,j} d_i d_j}}{(q^2;q^2)_{d_1}\cdots(q^2;q^2)_{d_m}} x_1^{d_1}\cdots x_m^{d_m}$$

# Quiver representation theory

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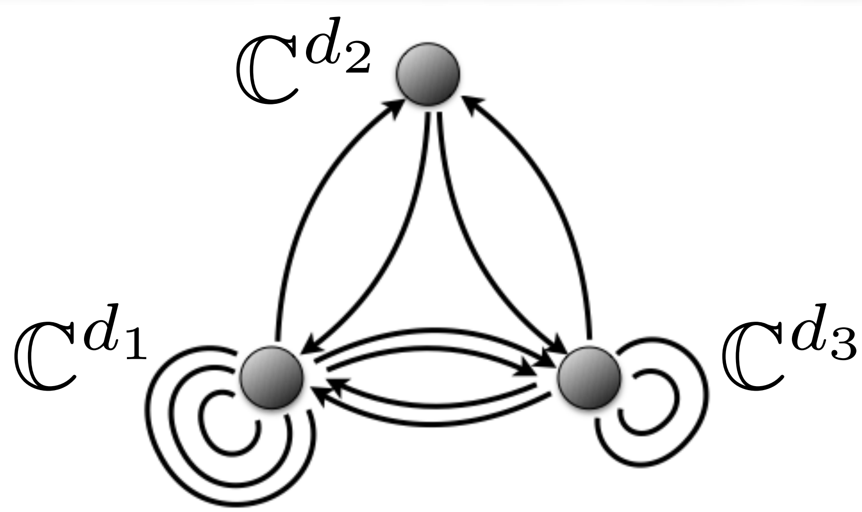
# Quiver representation theory

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**DT**

Recall, for a knot: $\sum_{r=0}^{\infty} x^r P_r(a,q) = \prod_{r,i,j,k} \left(1 - x^r a^i q^{j+2k+1}\right)^{OV_{r,i,j}}$

**LMOV**

**colored HOMFLY**

# Knots-quivers correspondence

With appropriate identification of variables, all colored HOMFLY polynomials are captured by the motivic generating function, for some particular symmetric matrix *C*:

$$P(x) = \sum_{r=0}^{\infty} P_r(a,q) x^r = \sum_{d_1,\ldots,d_m \geq 0} q^{\sum_{i,j} C_{i,j} d_i d_j} x^{d_1+\ldots+d_m} \frac{\prod_{i=1}^m q^{l_i d_i} a^{a_i d_i} (-1)^{t_i d_i}}{\prod_{i=1}^m (q^2;q^2)_{d_i}}$$

$$x_i = x a^{a_i} q^{l_i - 1} (-1)^{t_i}$$

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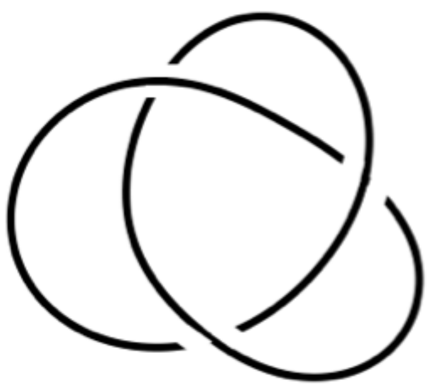
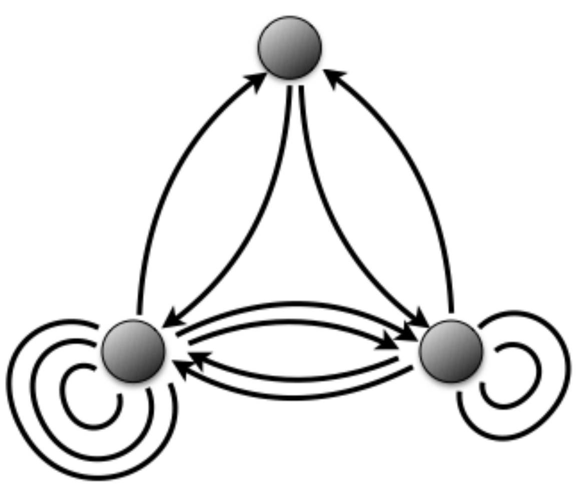
| Algebra of BPS states | Cohom. Hall Algebra |

# Example – trefoil knot

Colored polynomial for trefoil: $P_r(a,q) = \frac{a^{2r}}{q^{2r}} \sum_{k=0}^{r} \begin{bmatrix} r \\ k \end{bmatrix} q^{2k(r+1)} \prod_{i=1}^{k} (1 - a^2 q^{2(i-2)})$



We find: $C^{T_{2,3}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$



# Examples – torus knots

$$C^{T_{3,4}} = \begin{bmatrix} 0 & 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 3 & 5 \\ 2 & 3 & 4 & 4 & 5 \\ 3 & 3 & 4 & 4 & 5 \\ 5 & 5 & 5 & 5 & 6 \end{bmatrix}$$

$$C^{T_{3,5}} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 7 \\ 1 & 2 & 3 & 3 & 5 & 5 & 7 \\ 2 & 3 & 4 & 4 & 5 & 5 & 7 \\ 3 & 3 & 4 & 4 & 6 & 5 & 7 \\ 4 & 5 & 5 & 6 & 6 & 6 & 7 \\ 5 & 5 & 5 & 5 & 6 & 6 & 7 \\ 7 & 7 & 7 & 7 & 7 & 7 & 8 \end{bmatrix}$$

$$C^{T_{3,7}} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 5 & 6 & 7 & 8 & 9 & 11 \\ 1 & 2 & 3 & 3 & 5 & 5 & 6 & 7 & 7 & 9 & 9 & 11 \\ 2 & 3 & 4 & 4 & 5 & 6 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 3 & 4 & 4 & 5 & 5 & 7 & 7 & 7 & 9 & 9 & 11 \\ 4 & 5 & 5 & 5 & 6 & 6 & 7 & 7 & 8 & 9 & 10 & 11 \\ 5 & 5 & 6 & 5 & 6 & 6 & 8 & 7 & 7 & 9 & 9 & 11 \\ 5 & 6 & 6 & 7 & 7 & 8 & 8 & 8 & 9 & 9 & 10 & 11 \\ 6 & 7 & 7 & 7 & 7 & 7 & 8 & 8 & 8 & 9 & 10 & 11 \\ 7 & 7 & 8 & 7 & 8 & 7 & 9 & 8 & 8 & 9 & 9 & 11 \\ 8 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 10 & 10 & 11 \\ 9 & 9 & 10 & 9 & 10 & 9 & 10 & 10 & 9 & 10 & 10 & 11 \\ 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 12 \end{bmatrix}.$$

# Examples – $6_2$ knot

$$C^{6_2} = \begin{bmatrix} -2 & -2 & -1 & -1 & -1 & -1 & 0 & -1 & 1 & 1 & 1 \\ -2 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 \\ -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 2 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 2 & 2 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 2 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \end{bmatrix}$$

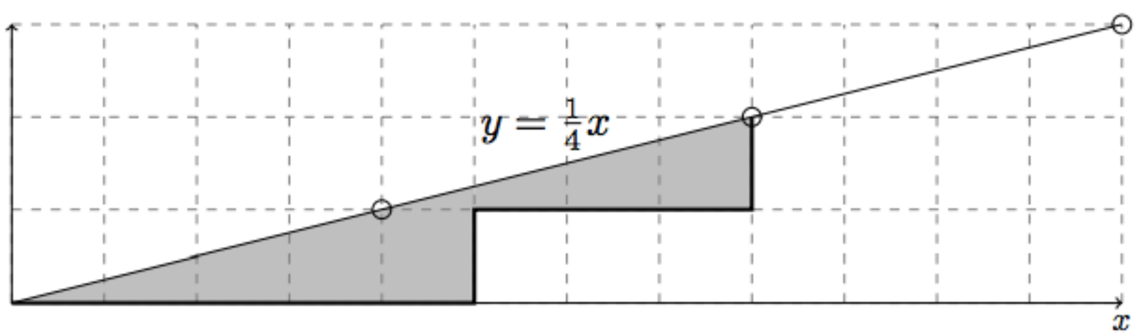
# Examples – $6_3$ knot

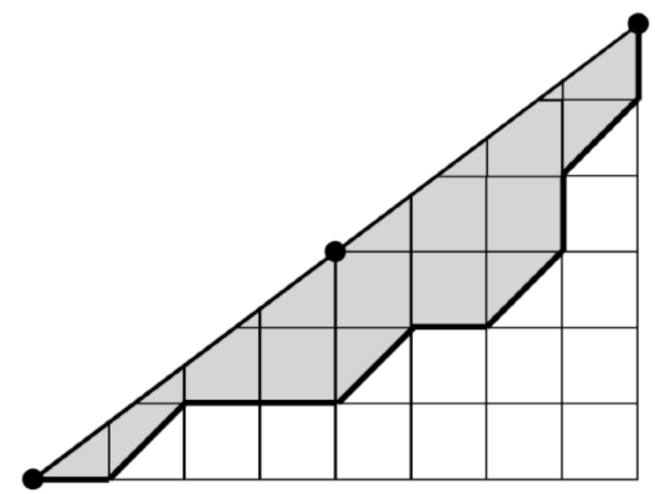
$$C^{6_3} = \begin{bmatrix} 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & -2 & 1 & 0 & -1 & -2 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -2 & 1 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -2 & -3 & 0 & -1 & -2 & -3 & -1 & 0 & -2 & -2 \\ -1 & -2 & -2 & -3 & -3 & -1 & -1 & -2 & -3 & -1 & -1 & -2 & -2 \\ 0 & 1 & 1 & 0 & -1 & 2 & 1 & 0 & -1 & 2 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & -1 & 2 & 1 & 1 & 0 \\ -1 & -1 & 0 & -2 & -2 & 0 & 0 & -1 & -2 & 0 & 0 & -1 & -2 \\ -1 & -2 & -2 & -3 & -3 & -1 & -1 & -2 & -2 & 0 & -1 & -1 & -2 \\ 0 & 1 & 1 & -1 & -1 & 2 & 2 & 0 & 0 & 3 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 & 1 & 0 & -1 & 2 & 2 & 1 & 0 \\ -1 & 0 & 0 & -2 & -2 & 1 & 1 & -1 & -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 0 & -2 & -2 & -1 & 0 & -2 & -2 & 0 & 0 & -1 & -1 \end{bmatrix}$$

# Further developments…

# Statistics of lattice paths

For torus knots, BPS numbers count lattice paths, which makes contact with combinatorics and models of statistical physics!



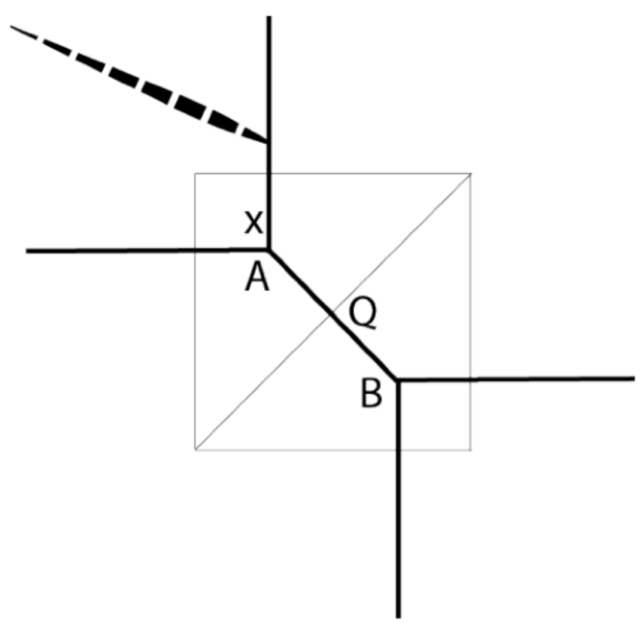



“Donaldson-Thomas invariants, torus knots, and lattice paths”
*Miłosz Panfil, Marko Stosic, P.S.*
*Phys. Rev. D98 (2018) 026022*

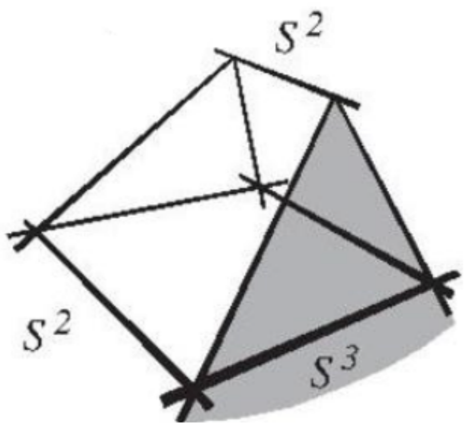
“Torus knots and generalized Schröder paths”
*Marko Stosic, P.S., arXiv: 2405.10161*

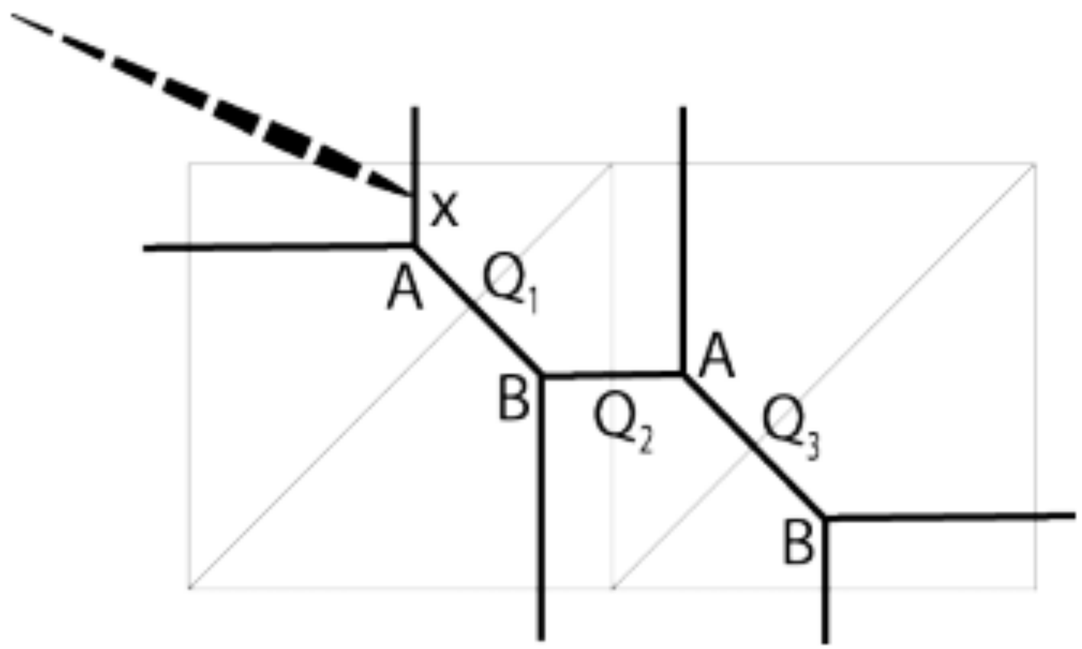
# Other Calabi-Yau manifolds

Analogous results arise for supersymmetric theories engineered by toric Calabi-Yau manifolds.




*Resolved conifold*




"Topological strings, strips and quivers"
*M. Panfil, P.S., JHEP 1901 (2019) 124*

*"Branes, quivers and wave-functions"*
*T. Kimura, M. Panfil, Y. Sugimoto, P.S.*
*SciPost Phys. 10 (2021) 051*

# Links with conformal field theory

$$P_C(x_1,\ldots,x_m) = \sum_{d_1,\ldots,d_m} \frac{(-q)^{\sum_{i,j=1}^m C_{i,j} d_i d_j}}{(q^2;q^2)_{d_1}\cdots(q^2;q^2)_{d_m}} x_1^{d_1}\cdots x_m^{d_m}$$

**"Fermionic form of VOA characters"**

"Fermionic sum representations for conformal field theory characters"
*R. Kadem, T. Klassen, B. McCoy, E. Melzer – Phys. Lett. B307 (1993) 68*

**"Nahm sums"**

"Conformal field theory and torsion elements of the Bloch group"
*W. Nahm – Les Houches School of Physics (2007)*

**Relations to logarithmic conformal field theories**

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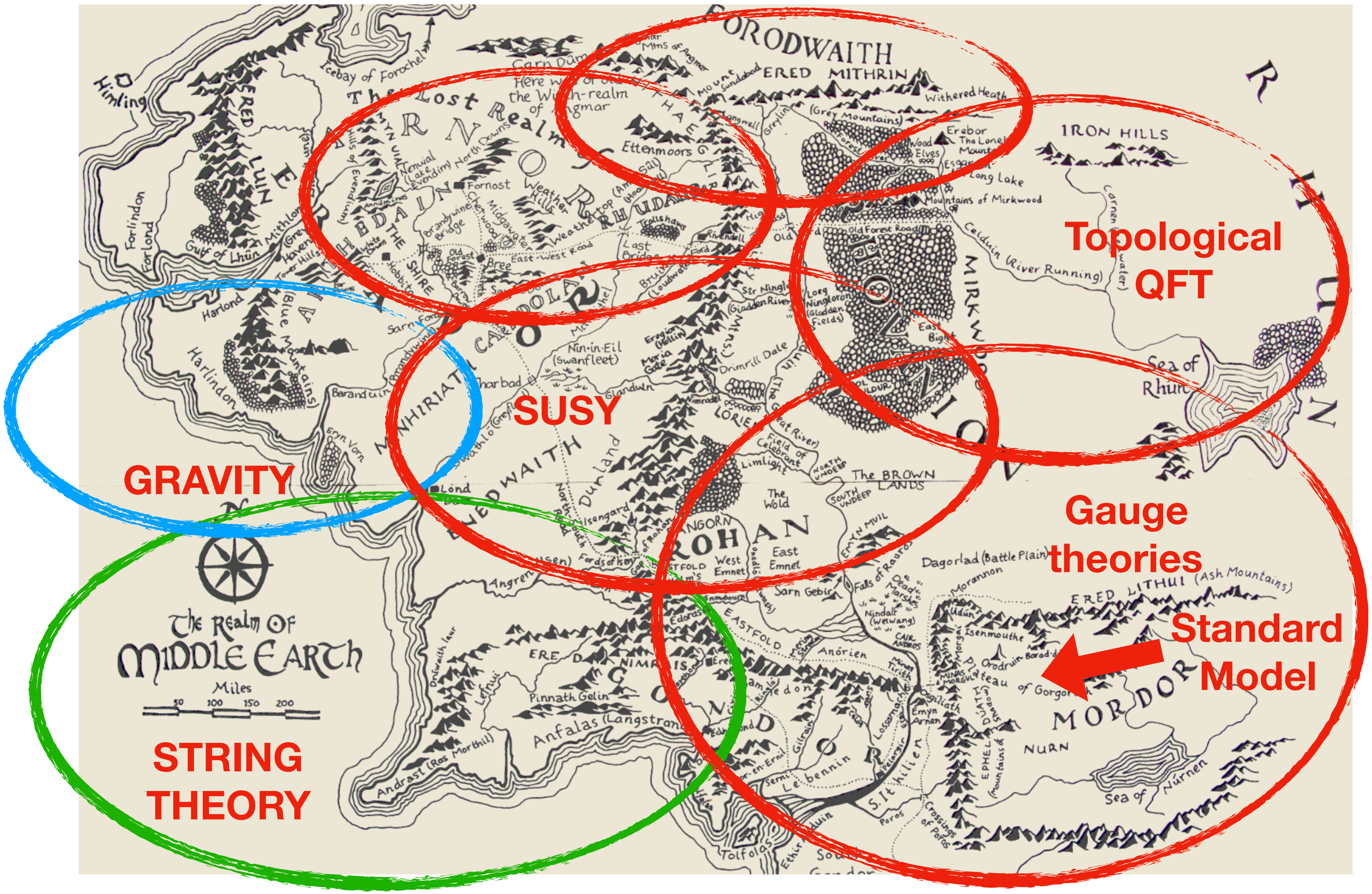
***Our group:*** **D.Bryan, L.Guerrini, P.Karimi, S.Kalinowski, J.Kenig, H.Potrykus, S.Purkayastha, A.Saha, P.S., ……..**





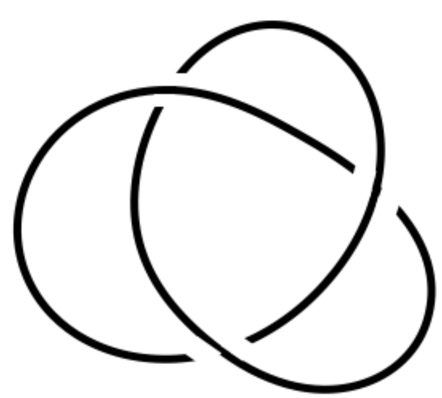

# Space of quantum field theories

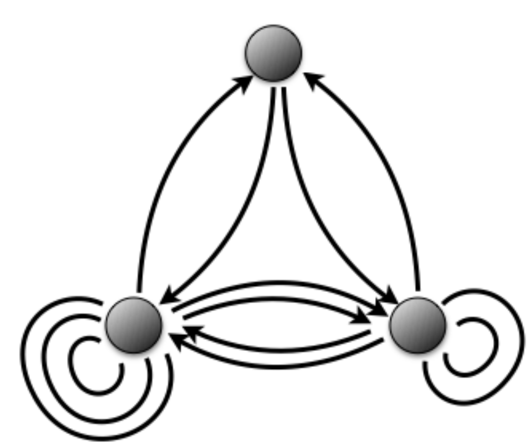


# Summary

(Hopefully) we learnt what are:

- Basic properties of quantum field theories with supersymmetry and topological invariance
- Geometric transition and other dualities motivated by string theory, which relate different quantum field theories
- Mathematical manifestation of such dualities in the form of knots-quivers correspondence
- Other relations to statistical models of lattice paths, conformal field theory, wave-function behavior for other Calabi-Yau systems,  etc.





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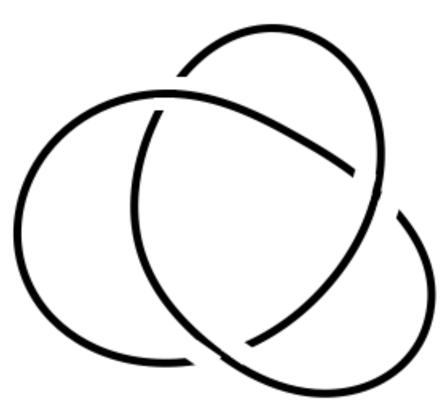
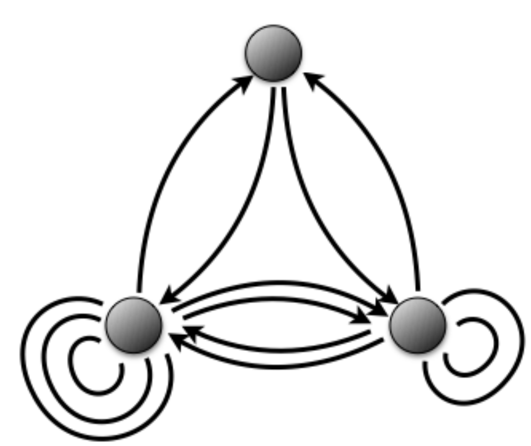
# Summary

- What else (in the spirit of *physical mathematics*) can physics and mathematics learn from each other?
- The links to sophisticated mathematics arise from simplest examples of quantum field theories – what would then arise for more complicated and realistic quantum field theories…?
- What is quantum field theory? What is the space of such theories? Is this an ultimate formalism to describe Nature?





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